MATH 210A: HOMEWORK 4

Problem 31. Let D_8 be the group of isometries of a square.

(a) Show that D_8 is generated by two elements ρ and σ, with $\rho^4 = \sigma^2 = 1$ and $\sigma\rho\sigma = \rho^{-1}$.
(b) Draw the subgroup tree for D_8. What are the normal subgroups of D_8?
(c) Find two subgroups $K, H < D_8$ such that $K \triangleleft H$ and $H \triangleleft D_8$ but K is not normal in D_8.

Problem 32. Let $n \geq 1$. Define the dihedral group D_{2n} (similar to above) by

$$D_{2n} := \langle \rho, \sigma : \rho^n = \sigma^2 = 1, \sigma\rho\sigma = \rho^{-1} \rangle.$$

(a) Prove that D_{2n} is the group of isometries of the regular n-gon and has order $2n$. (Hint: you can embed D_{2n} into S_n. The specific embedding depends on the parity of n.)
(b) Determine the centre $Z(D_{2n})$.
(c) Determine all normal subgroups of D_{2n}.
(d) Prove that $D_6 \cong S_3$, but $D_{24} \ncong S_4$.

Problem 33. Let G be a finite group, and let $a \neq b \in G$ such that $a^2 = b^2 = 1$. Prove that the subgroup generated by a and b is isomorphic to a dihedral group D_{2n}.

Problem 34. Let $\operatorname{Inn} G$ denote the set of *inner automorphisms* of G, that is, the set of automorphisms of the form $g \mapsto aga^{-1}$ for some fixed $a \in G$.

(a) Prove that $\operatorname{Inn} G$ is a subgroup of $\operatorname{Aut} G$.
(b) Prove that $\operatorname{Inn} G$ is, in fact, a normal subgroup of $\operatorname{Aut} G$.
(c) Prove that every automorphism of S_3 is inner, and that $\operatorname{Aut} S_3 \cong S_3$.
(d) Find an automorphism of D_4 that is not inner.

Problem 35. Show that if $\varphi \in \operatorname{Aut} S_4$ and $\tau \in S_4$ is a transposition, then $\varphi(\tau)$ is also a transposition. By studying the action of φ on transpositions, show that every automorphism of S_4 is inner.

Problem 36. There is a natural group homomorphism $\varphi : G \to \operatorname{Inn}(G)$. Prove that it is surjective and determine its kernel.

Problem 37. Let G be a group and let $N \subset Z(G)$ be a subgroup of its centre.

(a) Show that $N \triangleleft G$.
(b) Prove that if G/N is cyclic, then G is abelian.
(c) Suppose that $\operatorname{Aut} G$ is cyclic. Prove that G is abelian. (Hint: use the isomorphism implied by Problem 36 and part (b).)

Problem 38.

(a) Determine the order of the automorphism group of $\mathbb{Z}^2$.
(b) Determine the order of the automorphism group of $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Problem 39. Consider the inclusion $\mathbb{Z} \to \mathbb{Q}$ as abelian groups under addition. Describe the quotient group $\mathbb{Q}/\mathbb{Z}$ and the structure of its subgroups.

Problem 40. For a group G, let $[G, G]$ denote the subgroup generated by all elements $ghg^{-1}h^{-1}$.

(a) Prove that $[S_n, S_n] = A_n$ for $n \geq 3$.
(b) Prove that $[GL_n(\mathbb{C}), GL_n(\mathbb{C})] = SL_n(\mathbb{C})$ for $n \geq 3$. Hint: find a way to describe the generators of $SL_n(\mathbb{C})$ rather than using the characterisation $\det(M) = 1$.
(c) Does that equality hold for general fields k that are not necessarily algebraically closed?