Problem 31. Let $D_8$ be the group of isometries of a square.
   (a) Show that $D_8$ is generated by two elements $\rho$ and $\sigma$, with $\rho^4 = \sigma^2 = 1$ and $\sigma\rho\sigma = \rho^{-1}$.
   (b) Draw the subgroup tree for $D_8$. What are the normal subgroups of $D_8$?
   (c) Find two subgroups $K, H < D_8$ such that $K \triangleleft H$ and $H \triangleleft D_8$ but $K$ is not normal in $D_8$.

Problem 32. Let $n \geq 1$. Define the dihedral group $D_{2n}$ (similar to above) by
   
   $D_{2n} := \langle \rho, \sigma : \rho^n = \sigma^2 = 1, \sigma\rho\sigma = \rho^{-1} \rangle$.
   
   (a) Prove that $D_{2n}$ is the group of isometries of the regular $n$-gon and has order $2n$.
      (Hint: you can embed $D_{2n}$ into $S_n$. The specific embedding depends on the parity of $n$.)
   (b) Determine the centre $Z(D_{2n})$.
   (c) Determine all normal subgroups of $D_{2n}$.
   (d) Prove that $D_6 \cong S_3$, but $D_{24} \ncong S_4$.

Problem 33. Let $G$ be a finite group, and let $a \neq b \in G$ such that $a^2 = b^2 = 1$. Prove that
   the subgroup generated by $a$ and $b$ is isomorphic to a dihedral group $D_{2n}$.

Problem 34. Let $\text{Inn} G$ denote the set of inner automorphisms of $G$, that is, the set of
   automorphisms of the form $g \mapsto aga^{-1}$ for some fixed $a \in G$.
   (a) Prove that $\text{Inn} G$ is a subgroup of $\text{Aut} G$.
   (b) Prove that $\text{Inn} G$ is, in fact, a normal subgroup of $\text{Aut} G$.
   (c) Prove that every automorphism of $S_3$ is inner, and that $\text{Aut} S_3 \cong S_3$.
   (d) Find an automorphism of $D_4$ that is not inner.

Problem 35. Show that if $\varphi \in \text{Aut} S_4$ and $\tau \in S_4$ is a transposition, then $\varphi(\tau)$ is also a
   transposition. By studying the action of $\varphi$ on transpositions, show that every automorphism of $S_4$ is inner.

Problem 36. There is a natural group homomorphism $\varphi : G \rightarrow \text{Inn}(G)$. Prove that it is
   surjective and determine its kernel.

Problem 37. Let $G$ be a group and let $N \subset Z(G)$ be a subgroup of its centre.
   (a) Show that $N \triangleleft G$.
   (b) Prove that if $G/N$ is cyclic, then $G$ is abelian.
   (c) Suppose that $\text{Aut} G$ is cyclic. Prove that $G$ is abelian. (Hint: use the isomorphism
      implied by Problem 36 and part (b).)

Problem 38.
   (a) Determine the order of the automorphism group of $\mathbb{Z}^2$.
   (b) Determine the order of the automorphism group of $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Problem 39. Consider the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ as abelian groups under addition. Describe
   the quotient group $\mathbb{Q}/\mathbb{Z}$ and the structure of its subgroups.
Problem 40. For a group $G$, let $[G, G]$ denote the subgroup generated by all elements $ghg^{-1}h^{-1}$.

(a) Prove that $[S_n, S_n] = A_n$ for $n \geq 3$.
(b) Prove that $[GL_n(\mathbb{C}), GL_n(\mathbb{C})] = SL_n(\mathbb{C})$ for $n \geq 3$. Hint: find a way to describe the generators of $SL_n(\mathbb{C})$ rather than using the characterisation $\det(M) = 1$.
(c) Does that equality hold for general fields $k$ that are not necessarily algebraically closed?