Throughout, let $H < G$ denote that $H$ is a subgroup of $G$.

**Problem 25.** Consider the functor $F : \text{Group} \to \text{Set}$ sending a group $G$ to the set of its subgroups $\{ H : H < G \}$.

(a) Prove that this is indeed a functor.
(b) Is $F$ representable? Why or why not?

**Problem 26.** Let $G$ be a cyclic group.

(a) Prove that $G$ is determined up to isomorphism by $|G|$.
(b) Describe all subgroups of $G$.
(c) Determine all automorphisms of $G$.

**Problem 27.** Recall that the *normaliser* of a subgroup $H < G$, denoted $N_G H$, is the set of all elements $g \in G$ such that $gHg^{-1} = H$.

(a) Prove that $N_G H$ is a subgroup for any $H < G$.
(b) Give an example of a group $G$, a subgroup $H$, and $g \in G$ so that $gHg^{-1} \subsetneq H$ (strict containment).
(c) Define the *normal closure* of a subgroup and show by example that $N_G H$ needn’t be equal to the normal closure of $H$.

**Problem 28.**

(a) Show that if $g^2 = e$ for all $g$ in a group $G$, then $G$ is abelian.
(b) Show that every subgroup of index 2 is normal.
(c) Let $|G|$ be finite, and let $p$ be the smallest prime dividing $|G|$. Show that every subgroup of index $p$ is normal. (Hint: you need to use group actions to solve this, so you may need to save it for later.)

**Problem 29.**

(a) Determine the order of the symmetric group $S_n$.
(b) Prove that $S_n$ is generated by the set of all transpositions $(i \, j)$. (If you T\TeX\ this, please include a space \text \ in between entries in a cycle.)
(c) Prove that in fact, $S_n$ may be generated by just the transpositions $(1 \, 2), (1 \, 3), \ldots, (1 \, n)$.
(d) Prove that $S_n$ may also be generated by $(1 \, 2)$ and the $n$-cycle $(1 \, 2 \, \cdots \, n)$.
(e) Prove that if $n$ is prime, $S_n$ may be generated by any transposition and the $n$-cycle $(1 \, 2 \, \cdots \, n)$. Give an example of a transposition that won’t generate $S_4$ along with $(1 \, 2 \, 3 \, 4)$.

**Problem 30.** Recall from Problem 29(b) that any element $\sigma \in S_n$ can be written as a product of transpositions.

(a) Show that the number of transpositions in such an expression is always odd or always even.
(b) Use this to define the *signature map* $\text{sgn} : S_n \to C_2 = \{ \pm 1 \}$. Prove that this is a group homomorphism (where the group operation on $C_2$ is multiplication). What is the kernel of this map?