MATH 210A: HOMEWORK 3

Throughout, let $H < G$ denote that H is a subgroup of G.

Problem 25. Consider the functor $F : \mathbf{Group} \to \mathbf{Set}$ sending a group G to the set of its subgroups $\{H : H < G\}$.

(a) Prove that this is indeed a functor.
(b) Is F representable? Why or why not?

Problem 26. Let G be a cyclic group.

(a) Prove that G is determined up to isomorphism by $|G|$.
(b) Describe all subgroups of G.
(c) Determine all automorphisms of G.

Problem 27. Recall that the *normaliser* of a subgroup $H < G$, denoted $N_G H$, is the set of all elements $g \in G$ such that $gHg^{-1} = H$.

(a) Prove that $N_G H$ is a subgroup for any $H < G$.
(b) Give an example of a group G, a subgroup H, and $g \in G$ so that $gHg^{-1} \subsetneq H$ (strict containment).
(c) Define the *normal closure* of a subgroup and show by example that $N_G H$ needn't be equal to the normal closure of H.

Problem 28.

(a) Show that if $g^2 = e$ for all g in a group G, then G is abelian.
(b) Show that every subgroup of index 2 is normal.
(c) Let $|G|$ be finite, and let p be the smallest prime dividing $|G|$. Show that every subgroup of index p is normal. (Hint: you need to use group actions to solve this, so you may need to save it for later.)

Problem 29.

(a) Determine the order of the symmetric group S_n.
(b) Prove that S_n is generated by the set of all transpositions $(i\,j)$. (If you TeX this, please include a space `\,` in between entries in a cycle.)
(c) Prove that in fact, S_n may be generated by just the transpositions $(1\,2), (1\,3), \ldots, (1\,n)$.
(d) Prove that S_n may also be generated by $(1\,2)$ and the n-cycle $(1\,2\,\cdots\,n)$.
(e) Prove that if n is prime, S_n may be generated by any transposition and the n-cycle $(1\,2\,\cdots\,n)$. Give an example of a transposition that won't generate S_4 along with $(1\,2\,3\,4)$.

Problem 30. Recall from Problem 29(b) that any element $\sigma \in S_n$ can be written as a product of transpositions.

(a) Show that the number of transpositions in such an expression is always odd or always even.
(b) Use this to define the *signature map* $\operatorname{sgn} : S_n \to C_2 = \{\pm 1\}$. Prove that this is a group homomorphism (where the group operation on C_2 is multiplication). What is the kernel of this map?