MATH 210A: HOMEWORK 2

Problem 12. One definition of a group is that it is a *connected groupoid*. A category is called a *groupoid* if every morphism is an isomorphism, and a category is *connected* if $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is nonempty for all $x, y \in \mathcal{C}$.

(a) Prove that every connected groupoid is equivalent to a groupoid with one object.
(b) Construct a functor $\mathbf{Group} \to \mathbf{Cat}$ which sends a group to the appropriate groupoid with one object.
(c) Prove that this functor preserves products.

Problem 13. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We can define a new category F/d for any $d \in \mathcal{D}$ in the following way: the objects are pairs (c, α), where $c \in \mathcal{C}$ is an object and $\alpha : F(c) \to d$ is a morphism in $\mathcal{D}$. A morphism $(c, \alpha) \to (c', \alpha')$ is a morphism $f : c \to c'$ in $\mathcal{C}$ such that $\alpha' \circ F(f) = \alpha$. This is called a *slice category over d*.

(a) Prove that this is in fact a category.
(b) Define the dual notion of a slice category *under d* for any $d \in \mathcal{D}$ (denoted d/F) and prove it is a category.
(c) Choose a specific functor $F : \mathcal{C} \to \mathcal{D}$ to show that the F/d and d/F needn't be equivalent.

Problem 14. Recall the definition of an additive category. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is an *additive functor*, that is, for any $x, y \in \mathcal{A}$, the induced functor $\mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$ is an abelian group homomorphism.

(a) Prove that an arbitrary functor F is additive if and only if it preserves biproducts. Recall that one of the axioms of an additive category is that finite biproducts exist.
(b) If $\mathcal{A}$ is an additive category, then the Yoneda embedding can be viewed as a functor $\mathcal{A} \to \mathbf{Ab}^{\mathcal{A}}$, not just $\mathbf{Set}^{\mathcal{A}}$. Reinterpret the Yoneda lemma (both the covariant and contravariant versions) in this context.

Problem 15. Let $\mathcal{C}$ be a category and A a small category (that is, a category such that $\mathrm{Ob}\, A$ is a set). Let $F : A \to \mathcal{C}$ be a functor.

(a) Prove that $\lim_{a \in A} F(a)$ exists, then there is a canonical isomorphism
$$\mathrm{Mor}_{\mathcal{C}}(c, \lim_{a \in A} F(a)) \cong \lim_{a \in A}(c, F(a))$$
for any $c \in \mathcal{C}$.
(b) Prove that if $\mathrm{colim}_{a \in A} F(a)$ exists, then there is a canonical isomorphism
$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}_{a \in A} F(a), c) \cong \lim_{a \in A}(F(a), c)$$
for any $c \in C$.

Problem 16. Suppose that $L : \mathcal{C} \to \mathcal{D}$ admits a right adjoint. Prove that any two choices R_1, R_2 for this right adjoint are unique up to unique natural isomorphism.

Problem 17. Let $\mathcal{C}$ be a category, and let A be a small category. Define the *constant diagram* or *diagonal* functor $\Delta : \mathcal{C} \to \mathcal{C}^A$ by the following: for any $X \in \mathcal{C}$, let $\Delta(X)$ be the functor sending any $a \in A$ to X, and any morphism $f : a \to b$ in A to the morphism id_X.

(a) Prove that Δ is a functor.
(b) Assume that Δ admits a left adjoint. Prove that this functor is canonically isomorphic to the colimit functor $\mathrm{colim}_A : \mathcal{C}^A \to \mathcal{C}$.
(c) Make an analogous statement for the limit functor.

Problem 18. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *continuous* if it preserves all (small) limits that exist in $\mathcal{C}$, and *cocontinuous* if it preserves all (small) colimits that exist in $\mathcal{C}$. Prove that left adjoint functors are cocontinuous and that right adjoint functors are continuous.

Problem 19. An important concept in many categories is that of a 'free object'. In the general categorical context, we get a 'free' construction when the forgetful functor $U : \mathcal{C} \to \mathcal{D}$ admits a left adjoint. We saw this in Problem 6 on Homework 1 for the functor **Ring** $\to$ **Rng**.
(a) For a set X, describe the 'free topological space on X' (that is, what is the left adjoint to the forgetful functor **Top** $\to$ **Set**?).
(b) For a group G, describe the 'free ring on G'.
(c) Can you describe the 'free field' on any set X? Why or why not?

Problem 20. Although there is a left adjoint to the forgetful functor **Ab** $\to$ **Set** (a 'free' functor as above), prove there is no right adjoint.

Problem 21. Describe the free/forgetful adjunction from the category of abelian groups to the category of sets. Describe the unit and the counit of this adjunction. Is either a natural isomorphism?

Problem 22. Consider the category $\mathcal{C}$ of vector spaces (not necessarily finite dimensional) over a fixed field k. Consider the contravariant functor induced by the Yoneda embedding $h_k := \mathrm{Hom}_{\mathcal{C}}(-, k)$, viewing k as a k-vector space.
(a) Prove that h_k induces a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.
(b) Prove that $(h_k)^{\mathrm{op}}$ is right adjoint to h_k, and describe the counit and the unit of this adjunction. When is the unit an isomorphism?

Problem 23. Prove that for any small category $\mathcal{C}$ and any functor $F : \mathcal{C}^{\mathrm{op}} \to$ **Sets**, F can be written as a colimit of representable functors $h_x = \mathrm{Hom}_{\mathcal{C}}(-, x)$ coming from the Yoneda lemma.

Problem 24. Recall that we call a square

$$\begin{tikzcd} W \arrow[r, "f_1"] \arrow[d, "g_1"'] & X \arrow[d, "g_2"] \\ Y \arrow[r, "f_2"'] & Z \end{tikzcd}$$

a *pullback square* if W is the limit of the diagram $X \xrightarrow{g_2} Z \xleftarrow{f_2} Y$.
(a) Prove that if f_2 is a monomorphism, then so is f_1.
(b) Recall the definition of a *pushout square* and prove a dual result for epimorphisms: if g_1 is an epimorphism, then so is g_2.