MATH 210A: HOMEWORK 10

Problem 91. Let A be an abelian group.

(a) Show that A admits at most one structure of a left $\mathbb{Z}/n\mathbb{Z}$-module.
(b) Show that, for any multiplicative set $\Sigma \subset \mathbb{Z}$, that A admits at most one structure of a left $\mathbb{Z}[\Sigma^{-1}]$-module.

Problem 92. Let R be a ring.

(a) Let $I \subset R$ be a two-sided ideal. Show that there is an isomorphism of categories between left R/I-modules and left R-modules such that $I \cdot M = 0$.
(b) Let $\Sigma \subset R$ be a central multiplicative set. State and prove a statement analogous to the above about the category of left $R[\Sigma^{-1}]$-modules.

Problem 93. Let R be a commutative ring and M an R-(bi)module.

(a) Let $I \subset R$ be an ideal. Prove that the modules $R/I \otimes_R M$ and M/IM are isomorphic (as R-modules).
(b) Let $\Sigma \subset R$ be a multiplicative set. Prove that the modules $R[\Sigma^{-1}] \otimes_R M$ and $M[\Sigma^{-1}]$ are isomorphic (as R-modules).

Problem 94. Compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ for every $m, n \geq 1$.

Problem 95. Let k be a commutative ring and A and B two k-algebras, that is, two homomorphisms $k \to Z(A)$ and $k \to Z(B)$.

(a) Prove that the tensor product $A \otimes_k B$ has a ring structure such that the maps

$$A \to A \otimes_k B \quad a \mapsto a \otimes 1, \qquad B \to A \otimes_k B \quad b \mapsto 1 \otimes b$$

are both ring and k-module homomorphisms.
(b) Prove that $- \otimes_k -$ is the coproduct in the category of commutative k-algebras.

Problem 96. An abelian group A is called *divisible* if the endomorphism

$$m_n : A \to A \qquad a \mapsto n \cdot a$$

is surjective for any n. Prove that an abelian group is divisible if and only if it is an injective $\mathbb{Z}$-module.

Problem 97. A short exact sequence $L \overset{i}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} N$ is called *split* if it is isomorphic to a short exact sequence of the form $L \rightarrowtail L \oplus N \twoheadrightarrow N$ (with the usual maps). Prove that the following are equivalent.

(i) $L \overset{i}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} N$ is a split exact sequence.
(ii) The map $M \overset{p}{\twoheadrightarrow} N$ is a split epimorphism.
(iii) The map $L \overset{i}{\rightarrowtail} M$ is a split monomorphism.
(iv) There exists an isomorphism $\varphi : M \to L \oplus N$ so that $i \circ \varphi : L \to L \oplus N$ is the usual inclusion of L and $p \circ \varphi^{-1} : L \oplus N \to N$ is the usual projection onto N.
(v) There exists a retraction $r : M \to L$ of i and a section $s : N \to M$ of p such that $i \circ r + s \circ p = \mathrm{id}_M$.

Problem 98. Let R be a commutative ring. Prove that the following are equivalent for an R-module P.

(i) P is a projective module, i.e. the functor $\mathrm{Hom}(P, -)$ is exact.
(ii) For every morphism $f : P \to N$ and every epimorphism $\pi : M \to N$ there exists $g : P \to M$ such that $\pi\, g = f$.
(iii) Every short exact sequence $L \rightarrowtail M \twoheadrightarrow P$ splits.
(iv) P is the direct summand of a free module, i.e. there exists an R-module Q such that $P \oplus Q$ is free.

Problem 99. Let R be a commutative ring. Prove that the following are equivalent for an R-module I.

(i) I is an injective module, i.e. the functor $\mathrm{Hom}(-, I)$ is exact.
(ii) For every morphism $f : L \to I$ and every monomorphism $\iota : L \to N$ there exists $g : N \to I$ such that $g\, \iota = f$.
(iii) Every short exact sequence $I \rightarrowtail M \twoheadrightarrow N$ splits.

Problem 100. Let R be a commutative ring. An R-module M is called *flat* if the functor $M \otimes_R -$ is exact.

(a) Prove that projective modules are flat.
(b) Prove that, for any multiplicative set $\Sigma \subset R$, the localisation $R[\Sigma^{-1}]$ is flat as an R-module.
(c) Prove that a flat $\mathbb{Z}$-module must be torsion-free. (The converse is also true but trickier.)
(d) Prove that injective modules need not be flat by giving an explicit example in $\mathbb{Z}$-modules.