MATH 210A: HOMEWORK 1

Problem 1. Let X be a topological space. We can construct a category from X in the following way: the objects of this category are given by the open sets of X, and for two open sets U and V, the morphism set is defined by

$$\mathrm{Mor}(U,V) = \begin{cases} \{i : U \to V\} & \text{if } U \subset V \\ \varnothing & \text{else} \end{cases}.$$

(a) Show that this data, usually denoted $\mathrm{Open}(X)$, satisfies the axioms of a category.
(b) For any open subset $U \subset X$, let $\mathcal{F}(U)$ denote the set of continuous functions $U \to T$ for a fixed topological space T. Show that $\mathcal{F}$ defines a contravariant functor from $\mathrm{Open}(X)$ to the category of sets, usually denoted **Set**.
(c) Show that the assignment $X \mapsto \mathrm{Open}(X)$ induces a contravariant functor from the category of topological spaces **Top** to the category of (small) categories **Cat**.

Problem 2. Consider $\mathrm{Open}(X)$ as in the previous problem. Does $\mathrm{Open}(X)$ have an initial or final object? Does it admit products or coproducts? Explain why or why not, and if any of these exist, prove it.

Problem 3.

(a) A *monomorphism* in a category $\mathcal{C}$ is a morphism $f : A \to B$ that satisfies left cancellation: for any C ad any morphism $g : C \to A$, whenever $f \circ g = f \circ h$, then $g = h$. Show that the composition of two monomorphisms is a monomorphism, and determine the monomorphisms in **Set** and **Group**.
(b) Define the dual notion of an *epimorphism* and determine the epimorphisms in **Set** and **Group**.
(c) A *concrete category* $\mathcal{C}$ is one whose objects have underlying sets and whose morphisms are set morphisms with additional properties. This equivalently means that there is a faithful forgetful functor $U : \mathcal{C} \to$ **Set**. Prove that the map $i : \mathbb{Z} \to \mathbb{Q}$ in the category **Ring** is an epimorphism, but $U(i) : \mathbb{Z} \to \mathbb{Q}$ is not an epimorphism in **Set**.

Problem 4.

(a) Consider two 'parallel' morphisms $f, g : A \to B$ in a category $\mathcal{C}$. An *equalizer* of f and g is a morphism $h : C \to A$ such that $f \circ h = g \circ h$ and is satisfies a universal property: if $h' : C' \to A$ is another morphism such that $f \circ h' = g \circ h'$, then there exists a unique $\alpha : C' \to C$ such that $h \circ \alpha = h'$. Show that equalizers exist in **Set** and **Ab**.
(b) Define the dual notion of a *coequalizer* and show that these also exist in **Set** and **Ab**.
(c) Do equalizers and coequalizers exist in **Top**? Why or why not?

Problem 5. For each of the following categories, discuss the existence of initial objects, final objects, products, limits, coproducts, and colimits. It might be helpful to recall the definition of each in an arbitrary category before attempting this problem.

(a) **Set**, the category of sets;

(b) **Group**, the category of groups;
(c) **Ab**, the category of abelian groups;
(d) **Rng**, the category of rings (not necessarily unital). Henceforth we will call these 'rngs';
(e) **Ring**, the category of (unital) rings. In this category, we assume all morphisms $f : R \to S$ satisfy $f(1_R) = 1_S$;
(f) **Top**, the category of topological spaces.

Problem 6.

(a) Consider the categories **Rng** and **Ring** of rngs and rings (respectively). There is a forgetful functor $U : \mathbf{Ring} \to \mathbf{Rng}$ which forgets that a unital ring R has a unit. Show that U has a left adjoint $F : \mathbf{Rng} \to \mathbf{Ring}$ and prove that it is an adjoint.
(b) Show that an analogous result holds for the category of (unital) rings and the category of commutative rings.

Problem 7. Let $L : \mathcal{C} \to \mathcal{D}$ be a functor, left adjoint to $R : \mathcal{D} \to \mathcal{C}$. Show that if the counit $L \circ R \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, then R is fully faithful. Is the converse true?

Problem 8. Consider the category $\mathcal{C}_k$ whose objects are the natural numbers $\{0, 1, 2, \ldots\}$. For two objects m, n, we define $\mathrm{Mor}_{\mathcal{C}_k}(m, n) := M_{n,m}(k)$ for some fixed field k, the set of $n \times m$ matrices with entries in k. Prove that $\mathcal{C}_k$ is equivalent to the category of finite-dimensional vector spaces over k, with morphisms given by linear transformations.

Problem 9. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a morphism of categories with products. Show that, for $X, Y \in \mathcal{C}$, there is a natural morphism $F(X \times Y) \to F(X) \times F(Y)$. Show that if F has a left adjoint that this morphism is an isomorphism.

Problem 10. Consider the assignment $A \mapsto A^\times$ which sends a (unital) ring to its group of units.

(a) Show that this assignment induces a functor $F : \mathbf{Ring} \to \mathbf{Group}$.
(b) Show that this functor is representable, and find the appropriate ring which represents it.
(c) How many natural automorphisms of F are there? Recall that a natural automorphism is a natural transformation $\mu : F \Rightarrow F$ which is invertible.

Problem 11. A category $\mathcal{A}$ is called *preadditive* if each morphism set $\mathrm{Mor}_{\mathcal{A}}(X, Y)$ is equipped with the structure of an abelian group such that composition is a bilinear operation.

(a) Prove that an object is initial in a preadditive category if and only if it is final. Such an object is called a *zero object*.
(b) Assume that $\mathcal{A}$ admits finite products and finite coproducts (including a zero object, which we can view as an empty product or coproduct). Show that in a preadditive category, these notions coincide. That is, the canonical homomorphism $A_1 \sqcup \cdots \sqcup A_n \to A_1 \times \cdots \times A_n$ is an isomorphism. In this case, we call $\mathcal{A}$ an *additive category.*
(c) Consider the list of examples from Problem **??**. Which of these categories are additive, and why?