Math 210A Homework 9

Question 1. Let R be a commutative ring and let $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$.

(a) Prove that the polynomial f is nilpotent if and only if all a_i are nilpotent in R.

(b) Show that f is a zero divisor in $R[X]$, if and only if there exists a non-zero $b \in R$ such that $ba_n = \cdots = ba_1 = ba_0 = 0$.

(c) Show that f is invertible if and only if a_0 is invertible and all a_i are nilpotent for $i \geq 1$.

(d) If R is a domain, what is $(R[X])^\times$?

Question 2. Let R and S be two commutative rings.

(a) Let $\phi : R \to S$ be a ring homomorphism. Fix elements $s_1, \cdots, s_n \in S$. Show that the map $R[X_1, \cdots, X_n] \longrightarrow S$ given by $f(X_1, \cdots, X_n) \mapsto (\phi(f))(s_1, \cdots, s_n)$ is a ring homomorphism (sometimes called the *evaluation homomorphism*).

(b) Describe all ring homomorphisms from $R[X_1, \ldots, X_n] \longrightarrow S$.

(c) Prove that part (a) is false if we remove the commutative assumption.

Question 3.

(a) Let D be a domain and $c \in D$. Show that a polynomial $f(X) = \sum_{i=0}^n a_i X^i \in D[X]$ is irreducible in $D[X]$ if and only if $f(X - c) = \sum_{i=0}^n a_i (X - c)^i \in D[X]$ is irreducible.

(b) For each prime integer p, prove the cyclotomic polynomial $X^{p-1} + X^{p-2} + \cdots + X + 1$ is irreducible in $\mathbb{Z}[X]$. (Hint: Use Eisenstein and part (a).)

Question 4.

(a) Prove that $X^2 + Y^2 - 1$ is irreducible in $\mathbb{Q}[X, Y]$.

(b) Prove that $X^5 + Y^5 + Z^5$ is irreducible in $\mathbb{C}[X, Y, Z]$.

(c) Determine whether or not the polynomial $2X^5 - 6X + 6$ is irreducible in the following rings: $\mathbb{Z}[X]$, $(S^{-1}\mathbb{Z})[X]$ where $S = \{2^n \mid n \geq 0\}$, $\mathbb{Q}[X]$, $\mathbb{R}[X]$, and $\mathbb{C}[X]$.

(d) Determine all $n \in \mathbb{Z}$ such that the polynomial $X^3 + nX + 2$ is irreducible in $\mathbb{Q}[X]$.

Due date: Tuesday, December 2 (in discussion hour).