Math 210A Homework 8

Question 1. Equip a commutative ring R with the I-adic topology, for some ideal $I \subset R$.

(a) Show that a series $\sum_{i=0}^{\infty} x_i$ is Cauchy if and only if $\lim_{i \to \infty} x_i = 0$.

(b) Is (a) valid for any topological ring?

(c) Show that any element of the completion $\hat{R}$ is equal to a sum $\sum_{i=0}^{\infty} x_i$ with $x_i \in I^i$.

(d) Show that every p-adic integer $a \in \mathbb{Z}_p^{\wedge}$ can be written as the sum of an infinite series $a = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \ldots$ with $0 \leq a_i \leq p - 1$ for all $i \geq 0$.

(e) Given $a \in \mathbb{Z}_p^{\wedge}$ as in part (d), are the coefficients a_i unique?

(f) Compute $a \in \mathbb{Z}_p^{\wedge}$ as in part (d) when $a_i = p - 1$ for all $i \geq 0$.

(g) Show that for any $b \in \hat{I}^{(1)}$, the element $1 + b$ is invertible in $\hat{R}$. Describe $(1 + b)^{-1}$.

(h) Show that x is invertible in $\hat{R}$ if and only if $\pi_1(x) \in R/I$ is invertible.

(i) Determine when an element of $k[[X]]$ is invertible (for k a commutative ring).

(j) Are there continuous ring homomorphisms $\mathbb{Z}[[X]] \to \mathbb{R}$?

(k) Are there continuous ring homomorphisms $\mathbb{Z}_p^{\wedge} \to \mathbb{R}$?

Question 2. Find a solution of $x^2 = 10$ in the ring of 3-adic integers $\mathbb{Z}_3^{\wedge}$.

Question 3. Let R be a euclidean domain with "euclidean function" $\phi : R - \{0\} \to \mathbb{N}$.

(a) Show that one can replace ϕ by another euclidean function $\nu : R - \{0\} \to \mathbb{N}$ such that:

$$\nu(a) \leq \nu(ab) \qquad \text{for all} \quad b \in R - \{0\}. \tag{1}$$

This (unnecessary) condition is often included in the definition of euclidean.

(b) Assume that the euclidean function satisfies (1). Then show that $u \in R$ is invertible if and only if $\phi(u) = \phi(1)$.

Question 4. Show that the ring of Gaussian integers $\mathbb{Z}[i]$ is euclidean. What are the units in this ring?

Question 5. Let k a field. Show that $k[[X]]$ is principal. [Hint: Show more.]

Question 6. Let R be a principal domain and let $a, b \in \mathbb{R}$ be non-zero. Show that $Ra+Rb = R \cdot \gcd(a, b)$ and $Ra \cap Rb = R \cdot \operatorname{lcm}(a, b)$. Is this true in any UFD?

Question 7. Let R be a domain and $S \subset R$ be multiplicative with $0 \notin S$. Consider $S^{-1}R$.

(a) Prove that if R is a PID then $S^{-1}R$ is also a PID.

(b) Is the same true for UFD instead of PID?

Question 8. Let R be the ring $\{a+b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$, with norm $\mathcal{N}(a+b\sqrt{10}) = a^2-10b^2$.

(a) Show that $\mathcal{N}(u)\mathcal{N}(v) = \mathcal{N}(uv)$ for all $u, v \in R$ and that $\mathcal{N}(u) = 0$ if and only if $u = 0$.

(b) Prove that $u \in R$ is a unit if and only if $\mathcal{N}(u) = \pm 1$.

(c) Show that 2, 3, $4 + \sqrt{10}$ and $4 - \sqrt{10}$ and all irreducible elements of R.

(d) Use these elements to deduce that R is not a UFD.

Question 9. Let R be a ring.

(a) Let a be a nilpotent element of R. Prove that $1 + a$ is invertible in R.

(b) Suppose that u is invertible and a is nilpotent. What do you need to conclude that $u + a$ is invertible?

DUE DATE: WEDNESDAY NOVEMBER 26 (in class)!