Math 210A Homework 7

Question 1. Let R be a commutative ring, I an ideal of R. Define the *radical* of I, denoted $\sqrt{I}$, to be

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ } for\ some\ n > 0\}. \tag{0.1}$$

(a) Prove that $\sqrt{I}$ is an ideal of R containing I.

(b) Let I and J be ideals of R. Prove that $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(c) Prove that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

(d) Prove that $\sqrt{I} = R$ if and only if $I = R$.

Question 2. Let R be a commutative ring. We call $\mathcal{N} = \sqrt{(0)}$ the *nilradical* of R.

(a) We say a ring is *reduced* if the only nilpotent element is 0. Prove that $R/\mathcal{N}$ is reduced.

(b) Prove that if J is an ideal of R with R/J reduced, then $J \supset \mathcal{N}$.

(c) Adapt parts (a) and (b) to consider $\sqrt{I}$ instead of $\mathcal{N}$, and prove those statements.

Question 3. Let R be a ring. We say a central element $e \in R$ is *idempotent* if $e^2 = e$.

(a) What are the "trivial" idempotents?

(b) Show that if R has an idempotent e, then $R \cong Re \times R(1 - e)$ as rings. Conversely, if $R \cong R_1 \times R_2$, show that R has an idempotent yielding this decomposition.

(c) How many idempotents e are there in an integral domain?

(d) How many idempotents are there in $\mathbb{Z}/p^m\mathbb{Z}$, for p a prime and $m \geq 1$?

(e) How many idempotents are there in $\mathbb{Z}/n\mathbb{Z}$, for n an integer?

Question 4.

(a) Find an integer x between 10000 and 99999 such that x^2 and x have the same last five digits. [Hint: Write the integer x as $x = CHINA$.]

(b) How many solutions are there to (a)?

Question 5.

(a) Formalize the meaning of the following statement: "Inverting multiplicative subsets commutes with taking radicals." Prove it.

(b) Formalize the meaning of the following statement: "Inverting multiplicative subsets commutes with quotienting by ideals". Prove it.

Question 6. Let R be an integral domain with field of fractions F. Show that if R' is an integral domain with $R \subset R' \subset F$, then the field of fractions of R' is isomorphic to F. Give an example with $R \subsetneq R'$.

Question 7. Let R be a ring and $S \subset R$ be a central multiplicative subset.

(a) Prove that S consists of invertible elements if and only if $R \cong S^{-1}R$.

(b) Give necessary and sufficient conditions for $S^{-1}R$ to be non-zero.

(c) Give necessary and sufficient conditions for $R \longrightarrow S^{-1}R$ to be injective.

(d) Let $R = \mathbb{Z}/6\mathbb{Z}$ and $S = \{1, 2, 4\}$. Prove that S is multiplicative, and determine $S^{-1}R$.

Question 8. Let R be a ring and $S \subset R$ be a central multiplicative subset.

(a) For every ideal $I \subset R$ show that $S^{-1}I := \{\frac{a}{s} \mid a \in I,\ s \in S\}$ is an ideal of $S^{-1}R$.

(b) Show that every ideal of $S^{-1}R$ is of the form $S^{-1}I$ as above. Is the I unique?

(c) Show that the operation $I \mapsto S^{-1}I$ commutes with sum of ideals, intersection and product.

(d) Show that the localization of a principal domain is principal.

(e) Describe the ideals of $\mathbb{Z}[1/a] = S^{-1}\mathbb{Z}$ where $S = \{1, a, a^2, a^3, \ldots\}$.

(f) Describe the ideals of $\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$ where $S = \{a \in \mathbb{Z} \mid p \nmid a\}$ for p a prime.