Math 210A Homework 6

Question 1. Let G-Sets be the category of sets with a G-action (objects) and G-equivariant maps (morphisms). Let $T : \mathsf{Sets} \longrightarrow G\text{-}\mathsf{Sets}$ be the obvious "trivial G-action" functor.

(a) Prove that the functor $(-)^G : G\text{-}\mathsf{Sets} \longrightarrow \mathsf{Sets}$ where $Y^G = \{y \in Y \mid gy = y, \forall g \in G\}$ (the fixed points) is a right adjoint to T.

(b) Prove that $L : G\text{-}\mathsf{Sets} \longrightarrow \mathsf{Sets}$ defined by $L(Y) = G\backslash Y = \{Gy \mid y \in Y\} = Y/\sim$ where $y \sim gy, \forall\, g \in G$ (the set of orbits of Y under the G-action) defines a left adjoint to T.

Question 2. For G a group, let $G' = [G, G]$. Let $G^{(0)} = G$ and $G^{(n)} = (G^{(n-1)})'$ for $n \geq 1$.

(a) Show that G is solvable if and only if there is an n with $G^{(n)} = 1$.

(b) Show that G is solvable if and only if it has a *normal* tower with abelian subquotients.

(c) Is it true that any subnormal tower of a solvable group is automatically normal?

Question 3. For any group G, we define a chain of subgroups by setting $Z_0(G) = 1$, $Z_1(G) = Z(G)$, and $Z_{i+1}(G)$ to be the subgroup of G containing $Z_i(G)$ such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Such a chain is called an *upper central series*. A group G is said to be *nilpotent* if $Z_c(G) = G$ for some $c \in \mathbb{N}$.

(a) We call a normal tower $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_m = G$ *central* if $H_{i+1}/H_i \subset Z(G/H_i)$ for each i. Show that G is nilpotent if and only if it admits a central normal tower.

(b) In G nilpotent, is any normal tower necessarily central?

(c) Let $1 \to N \to G \to H \to 1$ be a *central* extension (i.e. $N \subset Z(G)$). Show that G is nilpotent if and only if N and H are, if and only if H is.

(d) Can one remove "central" in statement (c)?

(e) Construct the *lower central series* of G by setting $G_0 = G$, $G_1 = [G, G]$, and $G_{i+1} = [G, G_i]$. Prove that G is nilpotent if and only if there is a $c \in \mathbb{N}$ such that $G_c = 1$.

(f) Show that any p-group is nilpotent.

(g) Show that nilpotent implies solvable, but that the converse is false.

(h) Show that the cartesian product of a finite number of nilpotent groups is nilpotent.

Question 4. Let G be the set $\mathbb{Z}^3$ with group law $(a,b,c)(a',b',c') = (a+a', b+b'+ac', c+c')$. Is G finite? What is $Z(G)$? Is G abelian? Is G nilpotent? Is G solvable?

Question 5. Let $H = \mathbb{Z}$ and let $K = \mathbb{Z}/2\mathbb{Z}$. Discuss the structure of the group $H \rtimes_\phi K$ when ϕ is the nontrivial homomorphism $\phi : K \longrightarrow Aut(H)$. This group is known as D_∞, the *infinite dihedral group.* Prove that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty$.

Question 6. Let $G =< x, y, z \mid [x,y] = y,\ [y,z] = z,\ [z,x] = x >$. Prove that G is the trivial group. [Hint: Good luck!]

Question 7. Suppose G is a finite group of order n and for each k dividing n, G has a unique subgroup of order k. Prove that G is cyclic.

Question 8. Describe all ring homomorphisms from $\mathbb{R}$ to $\mathbb{R}$.

Question 9.

(a) Let $R = \mathbb{Z}$, $I_1 = 6\mathbb{Z}$ and $I_2 = 15\mathbb{Z}$. Show that the canonical map $R/(I_1 \cap I_2) \longrightarrow R/I_1 \times R/I_2$ is not surjective.

(b) Let n be an integer with prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Prove that

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \mathbb{Z}/p_2^{\alpha_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{\alpha_k}\mathbb{Z} \tag{0.1}$$

as rings. Use this to establish the following isomorphism on the groups of units

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})^\times \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})^\times . \tag{0.2}$$

(c) In the case that R is a non-commutative ring, prove that the statement of the Chinese Remainder Theorem is false by showing that comaximality of two two-sided ideals I and J is not enough to conclude that $I \cap J = IJ$.

Question 10. Let K be a field and V be a finite dimensional K-vector space. Let $R = \mathrm{End}_K(V)$. Show that any left ideal I in R is principal, i.e. of the form $R \cdot \alpha$ for some $\alpha \in R$. [Hint: Choose α of maximal rank in I and use matrix representations.]