Math 210A Homework 5

Question 1.

(a) Determine all subgroups of A_4. Show that A_4 has no subgroup of order 6.

(b) Determine all normal subgroups of A_4.

(c) Does there exist a nontrivial group action of A_4 on a set of two elements?

Question 2. Prove that every automorphism of S_3 is inner and $\mathrm{Aut}(S_3) \cong S_3$.

Question 3.

(a) Prove that every group of order p^2 (for p a prime) is abelian.

(b) Let G be a group with $|G| = p \cdot q$ for p and q two prime numbers, $q > p$ and $q \not\equiv 1 \pmod{p}$. Prove that $G \cong \mathbb{Z}/pq\mathbb{Z}$.

(c) Is there a group G such that $G/Z(G)$ has order 143?

Question 4. Prove that any group of order $2k$, where k is odd, has a normal subgroup of index 2. (Hint: Let G act on itself by left translation, and G has an element of order 2.)

Question 5. Classify all groups of order 154. (Hint: First use Question 4.)

Question 6.

(a) Exhibit all composition series for the quaternion group Q_8.

(b) Exhibit all composition series for the dihedral group D_8.

Question 7. Consider $G = \mathrm{Aut}_{\mathsf{Sets}}(\mathbb{N})$. For $\sigma \in G$, define as usual its *fixed set* by $\mathbb{N}^\sigma := \{a \in \mathbb{N} \,|\, \sigma(a) = a\}$ and let $M(\sigma) := \mathbb{N} - \mathbb{N}^\sigma$ be the set moved by σ.

(a) Show that $S_\infty := \{\sigma \in G \mid M(\sigma) \text{ is finite}\}$ is a normal subgroup of G.

(b) Prove that $S_\infty \simeq \mathrm{colim}_{n \in \mathbb{N}} S_n$ under explicit inclusions $S_n \hookrightarrow S_{n+1}$.

(c) Let A be the subgroup of S_∞ consisting of those σ that act as an even permutation on $M(\sigma)$. Prove that $A = \mathrm{colim}_{n \in \mathbb{N}} A_n$.

(d) Prove that A is an infinite simple group.

(e) Find an example of another infinite simple group. (Hint: Think of groups of matrices).

Question 8.

(a) Prove that there are no simple groups of order 132.

(b) Prove that there are no simple groups of order 6545.

(c) How many elements of order 7 must there be in a simple group of order 168?

Question 9.

(a) Find all finite groups that have exactly two conjugacy classes.

(b) Find all finite groups that have exactly three conjugacy classes.

Question 10. Let G be a finitely generated group. For each positive integer n, show there are only finitely many subgroups of index n.