Math 210A Homework 4

Question 1 (The Alternating Group)**.** We define the group A_n to be the kernel of the signature homomorphism $\mathrm{sgn} : S_n \to \{\pm 1\}$ and we call it the *alternating group on n letters*.

(a) Show that A_n is normal in S_n and compute its order.

(b) Show that $S_n = A_n \rtimes Z/2$.

(c) Show that, for $n \geq 3$, A_n is generated by the 3-cycles $(i\ j\ k)$ in S_n. (Hint: A_n is generated by products of 2 transpositions.)

(d) Show that A_n (for $n \geq 4$) and S_n (for $n \geq 3$) have trivial centers.

Question 2. Let G be a group. The *commutator* of two elements $g, h \in G$ is $[g, h] := g{\cdot}h{\cdot}g^{-1}{\cdot}h^{-1}$. The *commutator subgroup* $[G, G]$ is the subgroup generated by all commutators.

(a) Prove that, for $n \geq 3$, the commutator subgroup $[S_n, S_n] = A_n$. (Hint: Recall that A_n is generated by the 3-cycles.)

(b) Prove that, for $n \geq 3$, the commutator subgroup $[\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})] = \mathrm{SL}_n(\mathbb{C})$. (Hint: Recall that $\mathrm{SL}_n(\mathbb{C})$ is generated by the elementary matrices $E_{ij}(\lambda)$ which have 1's on the diagonal, λ in the i, j spot (where $i \neq j$), and 0 everywhere else.)

Question 3. Let $n \geq 2$ be an integer. Show that the only subgroup of index 2 in S_n is A_n.

Question 4. Let H and K be subgroups of finite index of a group G. Then $[G : H \cap K]$ is finite and $[G : H \cap K] \leq [G : H][G : K]$. Furthermore, $[G : H \cap K] = [G : H][G : K]$ if and only if $G = HK$.

Question 5.

(a) Show that Q_8 is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$.

(b) Recall that, D_8, the dihedral group of order 8 is a split extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$.

(c) Show that a non-abelian group G of order 8 is necessarily an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$.

(d) Show that a non-abelian group of order 8 is isomorphic to either D_8 or Q_8.

(e) Describe the isomorphism classes of groups of order 8.

Question 6 (The 5-Lemma)**.** Consider the following diagram of groups:

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "\cong"] & G_2 \arrow[r] \arrow[d, "\cong"] & G_3 \arrow[r] \arrow[d, "\alpha"] & G_4 \arrow[r] \arrow[d, "\cong"] & G_5 \arrow[d, "\cong"] \\
H_1 \arrow[r] & H_2 \arrow[r] & H_3 \arrow[r] & H_4 \arrow[r] & H_5
\end{tikzcd}$$

where each square commutes and each row is an exact sequence (ker = im at each group).

(a) Prove that the morphism $\alpha : G_3 \longrightarrow H_3$ is an isomorphism as well.

(b) Can we weaken the hypotheses on the (external) vertical homomorphisms?