# Math 210A Homework 3

**Question 1.** Let $\mathcal{C}$ be a category, $\mathcal{I}$ a small category and $F : \mathcal{I} \to \mathcal{C}$ a functor.

(a) Prove that $\lim\limits_{i \in \mathcal{I}} F(i)$ is characterized by the existence, for every $T \in \mathcal{C}$, of a natural bijection (of sets)

$$\mathrm{Mor}_{\mathcal{C}}(T, \lim_{i \in \mathcal{I}} F(i)) \cong \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(T, F(i)) .$$

(b) Similarly, characterize $\operatorname*{colim}\limits_{i \in \mathcal{I}} F(i)$ as follows: For every $T \in \mathcal{C}$, there is a natural bijection

$$\mathrm{Mor}_{\mathcal{C}}(\operatorname*{colim}_{i \in \mathcal{I}} F(i)\, ,\, T) \cong \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(F(i), T) .$$

**Question 2.** Let $A$ be a semigroup (that is, a set with an associative law $a \cdot b$)

(a) Suppose $A$ has a left identity element $e_L \in A$ (that is, $e_L \cdot a = a$ for each $a \in A$). Suppose further that each element $a \in A$ has a left inverse. Prove that $A$ is a group.

(b) Suppose now that $A$ has a left identity and every element has a right inverse. Is this enough to conclude that $A$ is a group?

**Question 3.** Let $G$ be a cyclic group.

(a) Describe all subgroups of $G$.

(b) Find all automorphisms of $G$.

**Question 4.**

(a) Show that if $g^2 = e$ for every $g$ in a group $G$, then $G$ is abelian.

(b) Show that every subgroup of index $p = 2$ is normal.

(c) Let $p$ be an odd prime. Find a group with a non-normal subgroup of index $p$.

(d) Prove that if $G$ is a finite group of even order, then $G$ contains an element $a$ such that $a \neq e$ but $a^2 = e$.

**Question 5.**

(a) Determine the order of the symmetric group $S_n$.

(b) Prove that $S_n$ is generated by all the transpositions.

(c) Prove that $S_n$ is, in fact, generated by the transpositions $(1,2),(1,3),...,(1,n)$.

(d) Prove that $S_n$ can be generated by the transposition $(1,2)$ and the $n$-cycle $(1,2,...,n)$.

**Question 6.** Let $D_8$ be the group of isometries of a square (distance-preserving bijections).

(a) Show that it is generated by two elements $\rho$ and $\sigma$ such that $\rho^4 = 1$, $\sigma^2 = 1$ and $\sigma\rho\sigma = \rho^{-1}$.

(b) Determine all subgroups of $D_8$.

(c) Find subgroups $K \lhd H \lhd D_8$ such that $K$ is not normal in $D_8$.

**Question 7.** Let $n \geq 1$. Define a group by generators and relations as $D_{2n} = \langle \rho, \sigma \mid \rho^n = \sigma^2 = \sigma\rho\sigma\rho = 1\rangle$. It is called *the dihedral group of order* $2n$.

(a) Show that $D_{2n}$ indeed has order $2n$. (Hint: Embed $D_{2n}$ into $\mathrm{End}_{\mathsf{Ab}}(\mathbb{C})$.)

(b) Identify $D_{2n}$ as the isometries of the regular $n$-gone ($n \geq 3$).

(c) Determine the center $Z(D_{2n})$.

(d) Find all normal subgroups of $D_{2n}$.

(e) Prove that $D_6 \cong S_3$, but that $D_8 \ncong S_4$.

**Question 8.** Let $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ be the subgroup of *inner automorphisms* of $G$ (that is, automorphisms of the form $a \mapsto gag^{-1}$ for some $g \in G$). Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

**Question 9.**

(a) Let $G$ be a group, and let $N$ be a subgroup of the center $Z(G)$. Show that $N$ is normal in $G$. Prove that if $G/N$ is cyclic then $G$ is abelian.

(b) Let $G$ be a group and suppose $\mathrm{Aut}(G)$ is cyclic. Prove that $G$ is abelian. (Hint: Use the group $\mathrm{Inn}(G)$, defined in Question 8 and compare with part (a) above for $N$ maximal.)

**Question 10.** Show that a group with no non-trivial automorphism is trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hint: First check it is abelian and 2-torsion.