Math 210A Homework 2

Question 1. Let $\mathcal{C}$ be a category.

(a) Let X be a fixed object in $\mathcal{C}$. Consider a new category $\mathcal{C} \downarrow X$ where the objects are the morphisms $f : Y \longrightarrow X$ (for $Y \in \mathrm{Ob}(\mathcal{C})$) and morphisms between $f : Y \longrightarrow X$ and $g : Z \longrightarrow X$ is a morphism $h : Y \longrightarrow Z$ such that $gh = f$. Show that $\mathcal{C} \downarrow X$ is a category. This is sometimes called the *comma category of* $\mathcal{C}$ *over* X.

(b) Let $\mathcal{C} \downarrow X$ be as above. Prove that the product of two objects $f : Y \longrightarrow X$ and $g : Z \longrightarrow X$ in $\mathcal{C} \downarrow X$ is just the fiber product of Y and Z over X in $\mathcal{C}$. Explicitly describe the fiber product ("pull-back") in the categories Sets, Gps and Ab, if they exist.

Question 2. Dualize the situation in Question 1 to define a category $X \downarrow \mathcal{C}$ and the notion of a "push-out". Do push-outs exist in Sets, Gps, and Ab?

Question 3. (a) Let A and B be two abelian groups. Let $\mathrm{Hom}(A, B)$ be the set of group homomorphisms from A to B. Prove that $\mathrm{Hom}(A, B)$ is also an abelian group.

(b) Adapt the covariant and contravariant Yoneda Lemmas to this situation.

Question 4. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, and let G_1 and G_2 be two functors that are right adjoint to F. Prove there is a unique isomorphism of functors $G_1 \cong G_2$.

Question 5. Let F and G be two adjoint functors. Prove that the unit and counit are isomorphisms if and only if F and G are equivalences of categories.

Question 6. Find a left adjoint functor to the inclusion functor from Ab to Gps.

Question 7.

(a) Determine the "free commutative ring" over a given set E.

(b) Determine the "free ring" over a given rng R.

(c) Determine the "free topological space" over a given set E.

Question 8. Is there a right adjoint functor to the forgetful functor $F : \mathsf{Sets} \longrightarrow \mathsf{Top}$?

Question 9. Let $\mathcal{C}$ be a category, and $\mathcal{I}$ a small category. Define the *constant diagram functor* $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ as follows. For any object $X \in \mathrm{Ob}(\mathcal{C})$, define $\Delta(X) : \mathcal{I} \longrightarrow \mathcal{C}$ to be the constant functor taking every object of $\mathcal{I}$ to X and each morphism to id_X.

(a) Explain how this defines a functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{I}}$.

(b) Prove that if every object $F \in \mathcal{C}^{\mathcal{I}}$ has a colimit, then we can define a functor colim : $\mathcal{C}^{\mathcal{I}} \longrightarrow \mathcal{C}$ which is left adjoint to Δ. Is the converse true?

(c) Prove that if every object $F \in \mathcal{C}^{\mathcal{I}}$ has a limit, then we can define a functor $\lim : \mathcal{C}^{\mathcal{I}} \longrightarrow \mathcal{C}$ which is right adjoint to Δ. Is the converse true?

Question 10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. (Suppose, if you want, that all limits and colimits exist in both $\mathcal{C}$ and $\mathcal{D}$.) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be left adjoint to $G : \mathcal{D} \longrightarrow \mathcal{C}$. Prove that F preserves colimits and that G preserves limits.

Question 11. Express all the axioms of a group using only diagrams (in Sets).