Math 210A Homework 1

Question 1. Let X be a topological space. Define the category $\mathrm{InclTop}(X)$ – for *inclusion topology* – as follows. Let the objects be open subsets $U \subset X$ and let the morphism sets be

$$\mathrm{Mor}(U,V) \;=\; \begin{cases} \emptyset & \text{if } U \not\subset V \\ \{\mathrm{incl}_{VU}\} & \text{(one-point set) if } U \subset V. \end{cases} \tag{0.1}$$

(a) Check that $\mathrm{InclTop}(X)$ is indeed a category.

(b) Let M be another topological space, and define $\mathcal{F}(U)$ to be the set of continuous maps from U to M. Show that $\mathcal{F}$ defines a contravariant functor from the category $\mathrm{InclTop}(X)$ to the category of sets. (Such a functor is called a *presheaf of sets on* X.)

Question 2. Show that the construction $\mathrm{InclTop}(-)$ of Question 1 induces a contravariant functor from the category Top of topological spaces to the category of small categories.

Question 3. Let $\mathcal{C}$ be a category. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is called a *monomorphism* if for any two morphisms $g, h : Z \longrightarrow A$ the equality $fg = fh$ implies $g = h$.

(a) Show that the composition of two monomorphisms is a monomorphism. Determine the monomorphisms in the category of sets and in the category of groups.

(b) Suppose that all morphism sets $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ are abelian groups and that composition is bilinear ($\mathcal{C}$ is sometimes called *pre-additive*). Suppose that the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

is cartesian (a pull-back). Show that if f is a monomorphism then so is f'.

(c) Define the dual notion of an epimorphism in $\mathcal{C}$. Show that the composition of two epimorphisms is an epimorphism. Determine the epimorphisms in the category of sets and in the category of groups.

(d) Suppose f is an isomorphism in $\mathcal{C}$. Prove that f is both a monomorphism and an epimorphism.

Question 4. Consider the category of rings with ring homomorphisms (i.e., a function which respects addition, multiplication, and sends 1 to 1). Prove that the inclusion $\phi : \mathbb{Z} \longrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category of rings. Is it an isomorphism?

Question 5. Give two proofs of the second (contravariant) Yoneda Lemma given in class: First by giving a direct proof and second by using Yoneda for the opposite category.

Question 6. Determine the initial and final objects in the following categories:

(a) Sets of sets,

(b) Gps of groups,

(c) Ab of abelian groups,

(d) Rings of rings,

(e) Rngs of rings without unit,

(f) Top of topological spaces.

Question 7. For the categories of Question 6, discuss the existence of products, coproducts, limits and colimits.

Question 8. Consider a pair of "parallel" morphisms $f, g : X \longrightarrow Y$ in a category $\mathcal{C}$. An *equalizer* of f and g is an object Z and a morphism $h : Z \longrightarrow X$ such that $fh = gh$ and such that for every morphism $i : T \longrightarrow X$ such that $fi = gi$ there exists a unique morphism $j : T \longrightarrow Z$ such that $hj = i$.

Show that equalizers exist in the category of sets and the category of abelian groups.

Question 9. Define the dual notion of co-equalizer and discuss it in the same examples.

Question 10. Show that the following are equivalent for a category $\mathcal{C}$.

(i) All limits exist in $\mathcal{C}$.

(ii) In $\mathcal{C}$, all products exist and an equalizer exists for every pair of parallel morphisms.

State and prove the dual result for colimits.