TENSOR TRIANGULAR CHOW GROUPS

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Abstract. We propose a definition of the Chow group of a rigid tensor triangulated category. The basic idea is to allow "generalized" cycles, with non-integral coefficients. The precise choice of relations is open to some fine-tuning.

Hypothesis 1. Let $\mathcal{K}$ be an essentially small tensor triangulated category. Let us assume that its triangular spectrum in the sense of [1], $\mathrm{Spc}(\mathcal{K}) = \big\{\,\mathcal{P} \subset \mathcal{K} \,\big|\, \mathcal{P} \text{ is prime}\,\big\}$, is a *noetherian* topological space, *i.e.* that every open of $\mathrm{Spc}(\mathcal{K})$ is quasi-compact. Let us also assume that $\mathcal{K}$ is *rigid*, as explained in [4] (or [2], where this property was called *strongly closed*). These hypotheses allow us to use the techniques of filtration of $\mathcal{K}$ by (generalized) dimension of the support.

Definition 2. As in [2, Def. 3.1], let us consider $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ a *dimension function*, meaning that $\mathcal{P} \subseteq \mathcal{Q} \implies \dim(\mathcal{P}) \leq \dim(\mathcal{Q})$, with equality in the finite range only if $\mathcal{P} = \mathcal{Q}$ (*i.e.* $\mathcal{P} \subseteq \mathcal{Q}$ and $\dim(\mathcal{P}) = \dim(\mathcal{Q}) \in \mathbb{Z}$ forces $\mathcal{P} = \mathcal{Q}$). Examples are the Krull dimension of $\overline{\{\mathcal{P}\}}$ in $\mathrm{Spc}(\mathcal{K})$, or the opposite of its Krull codimension. Assuming $\dim(-)$ is clear from the context, we shall use the notation

$$\mathrm{Spc}(\mathcal{K})_{(p)} := \big\{\,\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,\big|\, \dim(\mathcal{P}) = p\,\big\}\,.$$

Remark 3. In my opinion, there is nothing conceptually remarkable about the free abelian group on $\mathrm{Spc}(\mathcal{K})_{(p)}$. Therefore I propose another definition of p-dimensional cycles. This requires some preparation.

Definition 4. Recall from [3, § 4] that a rigid tensor triangulated category $\mathcal{L}$ is called *local* if $a \otimes b = 0$ implies $a = 0$ or $b = 0$. Conceptually, this means that $\mathrm{Spc}(\mathcal{L})$ is a local space. In that case, $\mathrm{Spc}(\mathcal{K})$ has a unique closed point $* := 0 \subset \mathcal{L}$, which is prime by assumption.

Example 5. For every tensor triangulated category $\mathcal{K}$ and for every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, the following tensor triangulated category is local in the above sense:

$$\mathcal{K}_{\mathcal{P}} := \big(\mathcal{K}/\mathcal{P}\big)^{\natural}.$$

We call $\mathcal{K}_{\mathcal{P}}$ the *local category at* $\mathcal{P}$. There is an obvious (localization) functor

$$q_{\mathcal{P}} \;:\; \mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{P} \hookrightarrow \mathcal{K}_{\mathcal{P}}\,,$$

which is the composition of Verdier localization and idempotent completion. (The category $\mathcal{K}_{\mathcal{P}}$ is also the colimit of the $\mathcal{K}(U)$ over those (quasi-compact) open subsets $U \subseteq \mathrm{Spc}(\mathcal{K})$ which contain $\mathcal{P}$. See more in [4, § 2.2] if necessary.)

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Definition 6. Assuming that $\mathcal{L}$ is local and that $\mathrm{Spc}(\mathcal{L})$ is noetherian, the open complement of the unique closed point $\{*\}$ in $\mathrm{Spc}(\mathcal{L})$ is quasi-compact, *i.e.* $\{*\}$ is a "Thomason (closed) subset". Under the classification of thick $\otimes$-ideals of $\mathcal{L}$, see [1], this one-point subset corresponds to the minimal non-zero subcategory

$$\mathrm{Min}(\mathcal{L}) := \mathcal{L}_{\{*\}} = \big\{\, a \in \mathcal{L} \,\big|\, \mathrm{supp}(a) \subseteq \{*\} \,\big\}\,.$$

These are the objects with minimal possible support (empty or a point).

Remark 7. Some comments are in order:

(1) This subcategory was called the subcategory of *finite-length* objects in [2] and denoted $\mathrm{FL}(\mathcal{L})$. As far as I know, there is no reason for objects of $\mathrm{Min}(\mathcal{L})$ to have finite-length (in the categorical sense that they admit a finite filtration with simple subquotients). The present notation, $\mathrm{Min}(\mathcal{L})$, is less biased towards commutative algebra and therefore probably preferable. It is however an interesting question to find some structure theorems about $\mathrm{Min}(\mathcal{L})$.
(2) As the previous comment suggests, if we take $\mathcal{L} = \mathrm{K}^{\mathrm{b}}(R\text{-}\mathrm{proj})$ the category of perfect complexes for R noetherian and local, then $\mathcal{L}$ is local and $\mathrm{Min}(\mathcal{L})$ is the subcategory of perfect complexes with finite-length homology.
(3) One can of course consider $\mathrm{Min}(\mathcal{L})$ even if $*$ is not Thomason but in that case it would just be the zero subcategory $0 = \mathcal{L}_{\varnothing}$.

Definition 8. Let $p \in \mathbb{Z}$. We define the group of *generalized p-cycles* to be

$$\mathrm{Z}_p(\mathcal{K}) := \bigoplus_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(p)}} K_0\big(\,\mathrm{Min}(\mathcal{K}_{\mathcal{P}})\big)\,,$$

where K_0 is the Grothendieck K-group (the quotient of the monoid of isomorphism classes $[a]$ of objects under $\oplus$, by the submonoid of the $[a] + [\Sigma b] + [c]$ for which there exists a distinguished triangle $a \to b \to c \to \Sigma a$).

Out of nostalgia for usual cycles, a generalized p-cycle can be written $\sum_{\mathcal{P}} \lambda_{\mathcal{P}} \cdot \mathcal{P}$ or $\sum_{\mathcal{P}} \lambda_{\mathcal{P}} \cdot \overline{\{\mathcal{P}\}}$, for some $\lambda_{\mathcal{P}} \in K_0\big(\,\mathrm{Min}(\mathcal{K}_{\mathcal{P}})\big)$. This is a purely notational choice. The non-trivial point is that we allow coefficients $\lambda_{\mathcal{P}}$ to live in other abelian groups than $\mathbb{Z}$, namely the Grothendieck groups of the minimal categories at every $\mathcal{P}$.

Example 9. Let X be a (topologically) noetherian scheme and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ the derived category of perfect complexes, whose spectrum $\mathrm{Spc}(\mathcal{K}) \cong X$ recovers the underlying space of X. Let $\dim(-)$ be the (opposite of the) Krull (co)dimension. Then we recover the usual p-dimensional (resp. $(-p)$-codimensional) cycles. Indeed, we have by Thomason that $\mathcal{K}_{\mathcal{P}} \cong \mathrm{K}^{\mathrm{b}}(\mathcal{O}_{X,x}\text{-}\mathrm{proj})$ if $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ corresponds to $x \in X$. The reason why integral coefficients suffice over regular schemes is that the group homomorphism defined by length of homology

$$K_0\big(\,\mathrm{Min}(\mathrm{K}^{\mathrm{b}}(\mathcal{O}_{X,x}\text{-}\mathrm{proj}))\big) \longrightarrow \mathbb{Z}\,,$$

is an isomorphism if X is regular (at x). However, in general, the left-hand group could be tricky. It is discussed for instance in [5].

Now to the relations. The point is that there might be several relations. The most flexible and most obvious one is the following.

Definition 10. For a (specialization) closed subset $Y \subset \mathrm{Spc}(\mathcal{K})$, we set $\dim(Y) = \sup\big\{\, \dim(\mathcal{P}) \,\big|\, \mathcal{P} \in Y \,\big\}$ and consider the filtration $\cdots \subset \mathcal{K}_{(p)} \subset \mathcal{K}_{(p+1)} \subset \cdots \subset \mathcal{K}$ by dimension of support

$$\mathcal{K}_{(p)} = \big\{\, a \in \mathcal{K} \,\big|\, \dim(\mathrm{supp}(a)) \le p \,\big\}\,.$$

By [2, Thm. 3.24], we have $\mathrm{Z}_p(\mathcal{K}) \cong K_0\big((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\big)$. We can define the *$p$-boundaries* $\mathrm{B}_p(\mathcal{K})$ as the image in $\mathrm{Z}_p(\mathcal{K})$ of $\mathrm{Ker}\,\big(K_0(\mathcal{K}_{(p)}) \to K_0(\mathcal{K}_{(p+1)})\big)$. In other words we have the diagram with exact rows

$$\begin{array}{ccccc}
\mathrm{Ker}(\iota) & \rightarrowtail & K_0(\mathcal{K}_{(p)}) & \xrightarrow{\ \iota\ } & K_0(\mathcal{K}_{(p+1)}) \\
\downarrow & & \downarrow & & \\
\mathrm{B}_p(\mathcal{K}) & \rightarrowtail & \mathrm{Z}_p(\mathcal{K}) & \twoheadrightarrow & \mathrm{CH}_p(\mathcal{K})
\end{array}$$

in which we define $\mathrm{CH}_p(\mathcal{K}) := \mathrm{Z}_p(\mathcal{K})/\,\mathrm{B}_p(\mathcal{K})$ to be the quotient of p-cycles by p-boundaries. These groups could be called the *(K-theoretic) Chow groups of p-cycles in $\mathcal{K}$*, with respect to the chosen dimension function dim.

Remark 11. The above $\mathrm{Ker}(\iota)$ is an *ad hoc* replacement for the maybe more natural image of $K_1(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)})$ by a connecting homomorphism. The reason is that triangulated categories do not behave well with higher K-theory. However, with this definition, it is not too hard to check that $\mathrm{CH}_p(\mathcal{K}) = \mathrm{CH}_p(X)$ when X is a regular scheme and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$.

It is however tempting to give another definition of p-boundaries, closer to the classical ideas of equivalence of p-cycles by means of divisors of functions on $(p+1)$-dimensional varieties. For this we need a preparation.

Lemma 12. *Recall Hypothesis 1. Let $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K})$ be a prime with $\dim(\mathcal{Q}) \in \mathbb{Z}$ finite. Let $a \in \mathcal{K}$ be an object such that $\mathrm{supp}(a) = \overline{\{\mathcal{Q}\}}$. Let $\alpha : a \to a$ be an endomorphism such that $\mathcal{Q} \notin \mathrm{supp}(\mathrm{cone}(a))$, meaning that $\mathrm{cone}(\alpha) \in \mathcal{Q}$, or equivalently that $\alpha : a \to a$ is an isomorphism at $\mathcal{Q}$, i.e. in $\mathcal{K}_{\mathcal{Q}}$. Then there exists finitely many $\mathcal{P} \in \mathrm{supp}\big(\mathrm{cone}(\alpha)\big)$ with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) - 1$. These $\mathcal{P}$ all are generic points of some irreducible components of $\mathrm{supp}(\mathrm{cone}(\alpha))$. Moreover, for every such $\mathcal{P}$, we have $q_{\mathcal{P}}(\mathrm{cone}(\alpha)) \in \mathrm{Min}(\mathcal{K}_{\mathcal{P}})$ where $q_{\mathcal{P}} : \mathcal{K} \to \mathcal{K}_{\mathcal{P}}$.*

Proof. Let $Z := \mathrm{supp}(\mathrm{cone}(\alpha))$. Since $Z \subseteq \mathrm{supp}(a) = \overline{\{\mathcal{Q}\}}$, every $\mathcal{P} \in Z$ belongs to $\overline{\{\mathcal{Q}\}}$, meaning $\mathcal{P} \subseteq \mathcal{Q}$ and in particular $\dim(\mathcal{P}) \leq \dim(\mathcal{Q})$. Since $\mathcal{P} = \mathcal{Q}$ is excluded by hypothesis, we must even have $\dim(\mathcal{P}) \leq \dim(\mathcal{Q}) - 1$ by Definition 2. Moreover, since $\mathrm{Spc}(\mathcal{K})$ is noetherian and spectral, there are finitely many $\mathcal{P}_1, \ldots, \mathcal{P}_n \in Z$ such that $Z = \overline{\{\mathcal{P}_1\}} \cup \cdots \cup \overline{\{\mathcal{P}_n\}}$ is the decomposition in irreducible components. By the previous discussion $\dim(\mathcal{P}_i) \leq \dim(\mathcal{Q}) - 1$. Now, if $\mathcal{P} \in Z$ satisfies $\dim(\mathcal{P}) = \dim(\mathcal{Q}) - 1$, it must belong to $\overline{\{\mathcal{P}_i\}}$ for some i, meaning $\mathcal{P} \subseteq \mathcal{P}_i$. This yields $\dim(\mathcal{Q}) - 1 = \dim(\mathcal{P}) \leq \dim(\mathcal{P}_i) \leq \dim(\mathcal{Q}) - 1$ which forces equality (of dimensions) and therefore equality of primes $\mathcal{P} = \mathcal{P}_i$ by Definition 2 again.

For the moreover part, let $\mathcal{P} \in \mathrm{supp}(\mathrm{cone}(\alpha))$ be such that $\dim(\mathcal{P}) = \dim(\mathcal{Q}) - 1$. Recall from [1, Prop. 3.6] that for any tensor triangular functor $F : \mathcal{K} \to \mathcal{L}$, we have $\mathrm{supp}_{\mathcal{L}}(F(b)) = \varphi^{-1}(\mathrm{supp}_{\mathcal{K}}(b))$ for every $b \in \mathcal{K}$, where $\varphi = \mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is the induced map on spectra. Also recall from [1, Prop. 3.11 and Cor. 3.14] that for $F = q_{\mathcal{P}} : \mathcal{K} \to \mathcal{K}_{\mathcal{P}}$, the map $\mathrm{Spc}(q_{\mathcal{P}})$ is a homeomorphism of $\mathrm{Spc}(\mathcal{K}_{\mathcal{P}})$ onto the subspace $\big\{\, \mathcal{P}' \in \mathcal{K} \,\big|\, \mathcal{P}' \supseteq \mathcal{P} \,\big\}$ of $\mathrm{Spc}(\mathcal{K})$. Combining those two, we can identify

$$\mathrm{supp}_{\mathcal{K}_{\mathcal{P}}}(q_{\mathcal{P}}(b)) = \mathrm{supp}_{\mathcal{K}}(b) \cap \big\{\, \mathcal{P}' \in \mathrm{Spc}(\mathcal{K}) \,\big|\, \mathcal{P}' \supseteq \mathcal{P} \,\big\}$$

for every $b \in \mathcal{K}$. Applying this to $b = \mathrm{cone}(\alpha)$ shows that every point $\mathcal{P}' \in \mathrm{supp}_{\mathcal{K}_{\mathcal{P}}}(q_{\mathcal{P}}(\mathrm{cone}(\alpha))$ can be identified with some $\mathcal{P}' \in \mathrm{supp}_{\mathcal{K}}(\mathrm{cone}(\alpha)) = Z$ such that $\mathcal{P}' \supseteq \mathcal{P}$, meaning that $\mathcal{P} \in \overline{\{\mathcal{P}'\}}$. But we have seen that $\mathcal{P}$ is the generic point of some irreducible component of Z, so $\mathcal{P} \in \overline{\{\mathcal{P}'\}} \subseteq Z$ forces $\mathcal{P}' = \mathcal{P}$. Back in $\operatorname{Spc}(\mathcal{K}_{\mathcal{P}})$, where $\mathcal{P}' = \mathcal{P}$ corresponds to the unique closed point $*$, we then get $\operatorname{supp}_{\mathcal{K}_{\mathcal{P}}}(q_{\mathcal{P}}(\operatorname{cone}(\alpha))) \subseteq \{*\}$ as wanted. □

Definition 13. Let $p \in \mathbb{Z}$ and let $\mathcal{Q} \in \operatorname{Spc}(\mathcal{K})_{(p+1)}$. Let $a \in \mathcal{K}$ such that $\operatorname{supp}(a) = \overline{\{\mathcal{Q}\}}$ and $\alpha : a \to a$ such that $\operatorname{cone}(\alpha) \in \mathcal{Q}$. Let us denote by

$$\operatorname{div}(\alpha) := \sum_{\mathcal{P} \in \operatorname{Spc}(\mathcal{K})_{(p)}} [q_{\mathcal{P}}(\operatorname{cone}(\alpha))] \cdot \mathcal{P}\,.$$

By Lemma 12, this is a well-defined element of $\mathrm{Z}_p(\mathcal{K})$.

We can define an alternative subgroup of *p-boundaries* $\mathrm{B}'_p(\mathcal{K})$ inside the group $\mathrm{Z}_p(\mathcal{K})$ of p-cycles, as the subgroup generated by the following subset

$$\left\{ \operatorname{div}(\alpha) \,\middle|\, (\mathcal{Q}, a, \alpha) \text{ as above} \right\}$$

which belongs to $\mathrm{Z}_p(\mathcal{K})$ by the above Lemma. Finally, we define the p-dimensional *Chow′ group* of $\mathcal{K}$ as the quotient

$$\mathrm{CH}'_p(\mathcal{K}) = \mathrm{Z}_p(\mathcal{K}) / \mathrm{B}'_p(\mathcal{K})\,.$$

Remark 14. There is yet another possible variant for the relations, where one would drop $\operatorname{supp}(a) = \overline{\{\mathcal{Q}\}}$ and allow endomorphisms $\alpha : a \to a$ to be any endomorphism such that $\dim(\operatorname{supp}(a)) = p+1$ and $\dim(\operatorname{supp}(\operatorname{cone}(\alpha))) = p$. A lemma very similar to Lemma 12 can be applied then. This yields an intermediate type of boundaries $\mathrm{B}''_p(\mathcal{K})$ and a group $\mathrm{CH}''_p(\mathcal{K})$.

It is easy to check that $\mathrm{B}'_p \subset \mathrm{B}''_p \subset \mathrm{B}_p$ since we have an exact triangle $a \overset{\alpha}{\to} a \to \operatorname{cone}(\alpha) \to \Sigma a$ in $\mathcal{K}_{(p+1)}$. Hence $\mathrm{CH}'_p(\mathcal{K}) \twoheadrightarrow \mathrm{CH}''_p(\mathcal{K}) \twoheadrightarrow \mathrm{CH}_p(\mathcal{K})$. It is not clear whether these definitions differ or not.

Remark 15. It seems that these three definitions coincide for $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ with X regular (say, smooth projective). Only further investigation will show which is the best definition, if they differ.

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