ENDOTRIVIAL REPRESENTATIONS OF FINITE GROUPS AND EQUIVARIANT LINE BUNDLES ON THE BROWN COMPLEX

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Abstract. We relate endotrivial representations of a finite group in characteristic p to equivariant line bundles on the simplicial complex of non-trivial p-subgroups, by means of weak homomorphisms.

Dedicated to Serge Bouc on the occasion of his 60th birthday

1. Introduction

Let G be a finite group, p a prime dividing the order of G and $\Bbbk$ a field of characteristic p. For the whole paper, we fix a Sylow p-subgroup P of G.

Consider the *endotrivial* $\Bbbk G$-modules M, *i.e.* those finite dimensional $\Bbbk$-linear representations M of G which are $\otimes$-invertible in the stable category $\Bbbk G-\mathrm{stab} = \Bbbk G-\mathrm{mod}\,/\,\Bbbk G-\mathrm{proj}$; this means that the $\Bbbk G$-module $\mathrm{End}_{\Bbbk}(M)$ is isomorphic to the trivial module $\Bbbk$ plus projective summands. The stable isomorphism classes of these endotrivial modules form an abelian group, $\mathrm{T}_{\Bbbk}(G)$, under tensor product. This important invariant has been fully described for p-groups in celebrated work of Carlson and Thévenaz [CT04, CT05]. Therefore, for general finite groups G, the focus has moved towards studying the relative version:

$$\mathrm{T}_{\Bbbk}(G,P) := \mathrm{Ker}\left(\mathrm{T}_{\Bbbk}(G) \to \mathrm{T}_{\Bbbk}(P)\right).$$

We connect this piece of modular representation theory to the equivariant topology of the *Brown complex* $\mathcal{S}_p(G)$ *of* p*-subgroups*, see [Bro75]. This G-space $\mathcal{S}_p(G)$ is the simplicial complex associated to the poset of nontrivial p-subgroups of G, on which G acts by conjugation. The study of $\mathcal{S}_p(G)$ is a major topic in group theory, centered around Quillen's conjecture [Qui78], which predicts that if $\mathcal{S}_p(G)$ is contractible then it is G-contractible, *i.e.* G admits a non-trivial normal p-subgroup. Here, we focus on the Picard group $\mathrm{Pic}^G(\mathcal{S}_p(G))$ of G-equivariant complex line bundles on $\mathcal{S}_p(G)$; see Segal [Seg68].

Our main result, Theorem 4.1, relates those two theories as follows (see Cor. 4.13):

1.1. **Theorem.** *Suppose $\Bbbk$ algebraically closed. Then there exists an isomorphism*

$$\mathrm{T}_{\Bbbk}(G,P) \simeq \ \mathrm{Tors}_{p'}\,\mathrm{Pic}^G(\mathcal{S}_p(G))$$

where $\mathrm{Tors}_{p'}\,\mathrm{Pic}^G(\mathcal{S}_p(G))$ *is the prime-to-p torsion subgroup of* $\mathrm{Pic}^G(\mathcal{S}_p(G))$.

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The left-hand abelian group $\mathrm{T}_{\Bbbk}(G,P)$ is always finite; see Remark 4.12. About the right-hand side, it is true for general finite G-CW-complex X that the group $\mathrm{Pic}^G(X)$ can be interpreted as an equivariant cohomology group, namely $\mathrm{H}^2_G(X,\mathbb{Z})$; in particular it is a finitely generated abelien group; see Remark 2.7. Some readers will consider Theorem 1.1 as the topological answer to the modular-representation-theoretic problem of computing $\mathrm{T}_{\Bbbk}(G,P)$.

Since its origin in [Bro75, Qui78], the space $\mathcal{S}_p(G)$ is related to the p-local study of G. Closer to our specific subject, Knörr and Robinson in [KR89] and Thévenaz in [Thé93] already exhibited interesting relations between modular representation theory and equivariant K-theory of $\mathcal{S}_p(G)$. The connection we propose here does not only relate *invariants* of both worlds but appears at a slightly deeper level, in that it connects actual objects. Indeed, in Construction 3.1, we build complex line bundles over $\mathcal{S}_p(G)$ from endotrivial representations of G. This construction then yields the isomorphism of Theorem 1.1. It would actually be interesting to see whether similar constructions exist for other classes of modular representations of G, beyond endotrivial ones.

The attentive reader will appreciate that modular representations of G live in positive characteristic whereas complex line bundles on the space $\mathcal{S}_p(G)$ are rather "characteristic zero" objects. This cross-characteristic connection is made possible thanks to the use of torsion elements and roots of unity. More precisely, we use in a crucial way the re-interpretation [Bal13] of the group $\mathrm{T}_{\Bbbk}(G,P)$ in terms of *weak P-homomorphisms*. Let us remind the reader.

1.2. **Definition.** Let K be a field – which will be either $\Bbbk$ or $\mathbb{C}$ in the sequel. A function $u : G \longrightarrow K^* = K - \{0\}$ is a *(K-valued) weak P-homomorphism* if

(WH 1) $u(g) = 1$ when $g \in P$.

(WH 2) $u(g) = 1$ if $P \cap P^g = 1$.

(WH 3) $u(g_2\, g_1) = u(g_2)\, u(g_1)$ if $P \cap P^{g_1} \cap P^{g_2 g_1} \neq 1$.

The name comes from (WH 3) which is a weakening of the usual homomorphism condition. We denote by $\mathrm{A}_K(G,P)$ the group of all weak P-homomorphisms from G to K^*, equipped with pointwise multiplication: $(uv)(g) = u(g)\, v(g)$.

The main result of [Bal13] is the existence of an explicit isomorphism

$$A_{\Bbbk}(G,P) \simeq \mathrm{T}_{\Bbbk}(G,P)\,. \tag{1.3}$$

This result has already found interesting applications, for instance the computation of $\mathrm{T}_{\Bbbk}(G,P)$ for new classes of groups by Carlson-Mazza-Nakano [CMN14] and Carlson-Thévenaz [CT15]. Here, we will use the complex version $A_{\mathbb{C}}(G,P)$ to build a homomorphism

$$\mathbb{L} : A_{\mathbb{C}}(G,P) \to \mathrm{Pic}^G(\mathcal{S}_p(G))$$

which will yield the isomorphism of Theorem 1.1 when suitably restricted to torsion. Injectivity of $\mathbb{L}$ on torsion relies in an essential way on a result of Symonds [Sym98], namely the contractibility of the orbit space $\mathcal{S}_p(G)/G$.

As often in such matters, it is difficult to predict which way traffic will go on the new bridge opened by Theorem 1.1. Computations of $\mathrm{T}_{\Bbbk}(G,P)$ have already been performed for many classes of finite groups and it seems quite possible that these examples will produce new equivariant line bundles for people interested in the G-homotopy type of $\mathcal{S}_p(G)$. Conversely, Theorem 1.1 might prove useful to modular representation theorists in endotrivial need. Only future work will tell.

Finally, we emphasize that the G-space $\mathcal{S}_p(G)$ can of course be replaced by any G-homotopically equivalent G-space, like Quillen's version [Qui78] via elementary abelian p-subgroups, Bouc's variant [Bou84], or Robinson's, see Webb [Web87].

2. The Brown complex and roots of functions

In this preparatory section, we gather some background and notation.

2.1. *Notation.* For an integer $m \geq 1$ and a field K (which will be $\Bbbk$ or $\mathbb{C}$), we denote by $\mu_m(K) = \left\{ \zeta \in K \,\middle|\, \zeta^m = 1 \right\}$ the group of m^{th} roots of unity in K.

2.2. *Notation.* The Brown complex $\mathcal{S}_p(G)$ is (the geometric realization of) the simplical complex with one non-degenerate n-simplex $[Q_0 < Q_1 < \cdots < Q_n]$ for each sequence of n proper inclusions of nontrivial p-subgroups, with the usual face-operations "dropping Q_i". For $n = 0$, we thus have a point $[Q]$ in $\mathcal{S}_p(G)$ for each non-trivial p-subgroup $Q \leq G$. The space $\mathcal{S}_p(G)$ admits an obvious *right G-action* given by conjugation on the p-subgroups, that is $Q \cdot g := Q^g = g^{-1}Qg$. This G-action on $\mathcal{S}_p(G)$ is compatible with the cell structure.

Since we have fixed a Sylow p-subgroup $P \leq G$, we can consider the subcomplex

$$Y := \mathcal{S}_p(P) \subseteq \mathcal{S}_p(G)$$

on those subgroups contained in P, *i.e.* we keep in Y those n-cells $[Q_0 < \cdots < Q_n]$ of $\mathcal{S}_p(G)$ corresponding to non-trivial subgroups of P. This closed subspace Y of $\mathcal{S}_p(G)$ is contractible, for instance towards the point $[P]$. But more than that, Y is an N-subspace of $\mathcal{S}_p(G)$ for $N = N_G(P)$ the normalizer of P. As such, Y is even N-contractible. See [TW91] if necessary. A fortiori, Y is P-contractible. The translates $Yg = \mathcal{S}_p(P^g)$ of the closed subspace Y cover the space $\mathcal{S}_p(G)$:

$$\mathcal{S}_p(G) = \cup_{g \in G}\ \mathcal{S}_p(P^g) = \cup_{g\in G}\ Yg.$$

We shall perform several "G-equivariant constructions" over $\mathcal{S}_p(G)$ by first performing a basic construction over Y and then showing that the translates of this basic construction on Yg_1 and on Yg_2 agree on the intersection $Yg_1 \cap Yg_2$ for all g_1, g_2.

2.3. *Remark.* We will be tacitly using the following fact. For $g_1, \ldots, g_n \in G$ (typically with $n \leq 3$), we have $P^{g_1} \cap \cdots \cap P^{g_n} \neq 1$ if and only if $Yg_1 \cap \cdots \cap Yg_n$ is not empty. Clearly a nontrivial $P^{g_1} \cap \cdots \cap P^{g_n}$ gives a point in $Yg_1 \cap \cdots \cap Yg_n$. Conversely, as G acts simplicially on $\mathcal{S}_p(G)$, a non-empty intersection $Yg_1 \cap \cdots \cap Yg_n$ must contain some 0-simplex $[Q]$, *i.e.* some nontrivial p-subgroup $Q \leq P^{g_i}$ for all i.

We shall also often use the following standard notation:

2.4. *Notation.* When $\lambda : L_1 \to L_2$ is a map of complex line bundles on a space X and $\epsilon : X \to \mathbb{C}^*$ is a continuous function, we denote by $\lambda \cdot \epsilon$ the map λ composed with the automorphism (of L_1 or L_2) which scales by $\epsilon(x)$ the fiber over x.

2.5. *Remark.* A G-equivariant complex line bundle L over a (right) G-space X consists of a complex line bundle $\pi : L \to X$ such that L is also equipped with a G-action making π equivariant and such that the action of every $g \in G$ on fibers $L_x \to L_{xg}$ is $\mathbb{C}$-linear. More generally, see [Seg68] for G-equivariant vector bundles. We denote by $\mathrm{Pic}^G(X)$ the group of G-equivariant isomorphism classes of such L, equipped with tensor product. The contravariant functor $\mathrm{Pic}^G(-)$ is invariant under G-homotopy. In particular, if X is G-equivariantly contractible, the map $\mathrm{Hom}_{\mathrm{gps}}(G, \mathbb{C}^*) \cong \mathrm{Pic}^G(*) \longrightarrow \mathrm{Pic}^G(X)$ is an isomorphism.

In the case of $X = \mathcal{S}_p(G)$, restriction to the P-subspace $Y = \mathcal{S}_p(P)$ yields a group homomorphism from $\mathrm{Pic}^G(\mathcal{S}_p(G))$ to the one-dimensional complex representations of P, that we shall simply denote by Res^G_P

$$\mathrm{Res}^G_P : \mathrm{Pic}^G(\mathcal{S}_p(G)) \to \mathrm{Pic}^P(\mathcal{S}_p(P)) \cong \mathrm{Hom}_{\mathrm{gps}}(P, \mathbb{C}^*). \tag{2.6}$$

2.7. *Remark* (Totaro). For a compact Lie group G acting on a manifold M, there is an isomorphism $\mathrm{Pic}^G(M) \simeq \mathrm{H}^2_G(M, \mathbb{Z}) = \mathrm{H}^2(M \times_G EG, \mathbb{Z})$, where $EG \to BG$ is the universal G-principal bundle on the classifying space BG; see [GGK02, Thm. C.47], where the similar result for a finite group acting on a finite CW-complex is attributed to [HY76]. Alternatively, one can see the latter by reducing to the case of manifolds, since every finite G-CW-complex is G-homotopy equivalent to a (non-compact) G-manifold. Then the group $\mathrm{H}^2(X \times_G EG, \mathbb{Z})$ can be approached via a Serre spectral sequence for the fibration $X \to X \times_G EG \to BG$. In particular, using that G is finite, the spectral sequence collapses rationally to an isomorphism $\mathrm{H}^2(X \times_G EG, \mathbb{Q}) \simeq \mathrm{H}^0(BG, \mathrm{H}^2(X, \mathbb{Q}))$ showing that $\mathrm{Pic}^G(X) \otimes \mathbb{Q} \simeq (\mathrm{Pic}(X) \otimes \mathbb{Q})^G$.

2.8. *Notation.* For a subspace Y of a G-space X, like our $Y = \mathcal{S}_p(P) \subseteq \mathcal{S}_p(G) = X$, every element $g \in G$ yields a homeomorphism $\cdot g : Y \overset{\sim}{\to} Yg$. We can transport things from Y to Yg via this homeomorphism, and we use $g_*(-)$ to denote this idea. For instance, if $f : Y \to \mathbb{C}$ is a function, then $g_* f : Yg \to \mathbb{C}$ is $g_* f(x) := f(xg^{-1})$. Another situation will be that of G-equivariant line bundles $L \overset{\pi}{\to} X$ and $L' \overset{\pi'}{\to} X$ and a morphism $\lambda : L_{|_Y} \to L'_{|_Y}$ of bundles over Y, in which case the morphism $g_*\lambda : L_{|_{Yg}} \to L'_{|_{Yg}}$ is defined by the commutativity of the following top face:

$$\begin{tikzcd}
L_{|_Y} \arrow[rr, "\cdot g", "\simeq"'] \arrow[dr, "\lambda"] \arrow[ddr, "\pi"'] & & L_{|_{Yg}} \arrow[dr, "=: g_*(\lambda)"] \arrow[ddr, "\pi"'] & \\
& L'_{|_Y} \arrow[rr, "\cdot g", "\simeq"'] \arrow[d, "\pi'"] & & L'_{|_{Yg}} \arrow[d, "\pi'"] \\
& Y \arrow[rr, "\cdot g", "\simeq"'] & & Yg\,.
\end{tikzcd} \tag{2.9}$$

As we use *right* actions (that is $(\cdot g_2 g_1) = (\cdot g_1) \circ (\cdot g_2)$) we have $(g_2 g_1)_* = (g_1)_* \circ (g_2)_*$.

Let us now say a word of roots of complex functions.

2.10. *Remark.* Throughout the paper, $\mathbb{C}$ is given the trivial G-action. Hence a G-map $f : X \to \mathbb{C}$ from a (right) G-space X to $\mathbb{C}$ is simply a continuous function such that $f(xg) = f(x)$ for all $x \in X$ and all $g \in G$, that is essentially a continuous function $\bar{f} : X/G \to \mathbb{C}$ on the orbit space.

2.11. **Proposition.** *Let $m \geq 1$ be an integer, X a G-space and $f : X \to \mathbb{C}^*$ a G-map. Suppose that f is G-homotopic to the constant map 1. Then f admits an m^{th} root in $\mathrm{Cont}_G(X, \mathbb{C}^*)$, i.e. a G-map $f^{1/m} : X \to \mathbb{C}^*$ such that $(f^{1/m})^m = f$.*

Proof. By assumption, the induced map $\bar{f} : X/G \to \mathbb{C}^*$ is homotopic to 1. Then it suffices to observe that $\bar{f}$ has an m^{th} root by a standard determination-of-the-logarithm argument. (Let $\bar{X} = X/G$ and let $H : \bar{X} \times [0,1] \to \mathbb{C}^*$ be a homotopy between $H(x,0) = \bar{f}(x)$ and $H(x,1) = 1$. Lifting each $t \mapsto H(x,t)/|H(x,t)| \in \mathbb{S}^1$ along the fibration $\mathbb{R} \twoheadrightarrow \mathbb{S}^1$, we find a map $\theta : \bar{X} \times [0,1] \to \mathbb{R}$ such that $H(x,t) = |H(x,t)| \cdot e^{i\theta(x,t)}$ and $\theta(x,1) = 0$. One can then define the m^{th} root of $\bar{f}$ via $\bar{f}^{1/m}(x) = |\bar{f}(x)|^{1/m} \cdot e^{i\theta(x,0)/m}$ for all $x \in \bar{X}$.) $\square$

2.12. **Corollary.** *If X/G is contractible (e.g. if X is G-contractible) then for every integer $m \geq 1$, every G-map $f : X \to \mathbb{C}^*$ admits an m^{th} root $f^{1/m} \in \mathrm{Cont}_G(X, \mathbb{C}^*)$.*

Proof. As such a map f factors via $X \twoheadrightarrow X/G$, the contractibility of X/G implies that f is G-homotopically trivial and we conclude by Proposition 2.11. □

2.13. **Corollary.** *For every integer $m \geq 1$, every G-map $f : \mathcal{S}_p(G) \to \mathbb{C}^*$ on the Brown complex admits an m^{th} root $f^{1/m} \in \mathrm{Cont}_G(\mathcal{S}_p(G), \mathbb{C}^*)$.*

Proof. The orbit space $\mathcal{S}_p(G)/G$ is contractible by Symonds [Sym98]. □

3. Constructing line bundles from weak homomorphisms

We now want to associate a G-equivariant complex line bundle L_u on $\mathcal{S}_p(G)$ to each complex-valued weak homomorphism $u \in A_{\mathbb{C}}(G, P)$ as in Definition 1.2. In essence, this is a very standard gluing procedure, familiar to every geometer. We spell out some details for the sake of clarity and to see where the "weak homomorphism" conditions (WH 1-3) show up.

3.1. **Construction.** Let $u : G \to \mathbb{C}^*$ be a weak P-homomorphism and $Y = \mathcal{S}_p(P) \subseteq \mathcal{S}_p(G)$ as in Notation 2.2. Define L_u as the following topological space:

$$\boxed{L_u := \Big(\bigsqcup_{s \in G} Ys \times \mathbb{C} \Big)\big/ \sim}$$

where $\sim$ is the equivalence relation defined in (3.2) below. We use the notation $(y, a)_s$ to indicate a point (y, a) in the space $Ys \times \mathbb{C}$ with index $s \in G$; and we shall write $[y, a]_s \in L_u$ for its class modulo $\sim$. (As the subsets Ys do intersect in $\mathcal{S}_p(G)$, the lighter notation (y, a) would be ambiguous.) Note that the weak P-homomorphism u does not appear so far; it is used in the equivalence relation:

$$(y, a)_s \sim (z, b)_t \qquad \text{iff} \qquad \begin{cases} y = z \\ \text{and} \\ a \cdot u(s\,t^{-1}) = b. \end{cases} \tag{3.2}$$

Direct inspection shows that $\sim$ is an equivalence relation: Reflexivity uses (WH 1); symmetry uses that $u(g^{-1}) = u(g)^{-1}$, see [Bal13, Rem. 4.2(1)]; transitivity relies on (WH 3) and Remark 2.3. Of course, L_u is equipped with the quotient topology.

3.3. *Remark.* A good way to keep track of what happens is to think of the class $[y, a]_s$ as a fictional element "$a \cdot s \in \mathbb{C}$ living in a fiber over $y \in \mathcal{S}_p(G)$", which is not defined since we do not know how $s \in G$ should act on $\mathbb{C}$. Still, equality between "$a \cdot s$ over y" and "$b \cdot t$ over z" should nonetheless mean that they live in the same fiber, *i.e.* $y = z$, and that "$a \cdot (st^{-1}) = b$". So we decide that the action of st^{-1}, *i.e.* the *difference* of the two actions over the point $y = z$ in $Ys \cap Yt$, is given via the weak homomorphism u. This can be compared to [Bal13, Eq. (2.7)].

The space L_u admits a continuous projection to the Brown complex

$$\pi_u : L_u \to \mathcal{S}_p(G)$$

simply given by $[y,a]_s \mapsto y$ and whose fibers are isomorphic to $\mathbb{C}$. More precisely, for every $s \in G$, we have a homeomorphism

$$\begin{aligned} \alpha_s : \qquad \mathbb{1}_{Ys} := Ys \times \mathbb{C} &\xrightarrow{\simeq} \pi_u^{-1}(Ys) \subseteq L_u \\ (y,a) &\longmapsto [y,a]_s \end{aligned} \tag{3.4}$$

(We denote trivial line bundles by $\mathbb{1}$.) These are *trivializations* of L_u over Ys. For all $s,t \in G$, the transition maps $\alpha_t^{-1}\alpha_s$ on the intersection

$$\begin{array}{ccccc} (Ys \cap Yt) \times \mathbb{C} & \xrightarrow[\simeq]{\alpha_s} & \pi_u^{-1}(Ys \cap Yt) & \xleftarrow[\simeq]{\alpha_t} & (Ys \cap Yt) \times \mathbb{C} \\ (y,a) & \longmapsto & [y,a]_s \overset{(3.2)}{=} [y, a \cdot u(s\,t^{-1})]_t & \longmapsto & (y, a \cdot u(s\,t^{-1})) \end{array}$$

is given by the (constant) linear isomorphism, multiplication by the unit $u(s\,t^{-1})$. In other words, $L_u \xrightarrow{\pi_u} \mathcal{S}_p(G)$ is a complex line bundle on $\mathcal{S}_p(G)$. We record the above computation in compact form: for all $s,t \in G$ we have an equality

$$\alpha_s = \alpha_t \cdot u(s\,t^{-1}) \qquad \text{over } Ys \cap Yt \tag{3.5}$$

as isomorphisms $\mathbb{1}_{Ys\cap Yt} \overset{\sim}{\to} (L_u)_{|Ys\cap Yt}$. Here we used Notation 2.4.

The right G-action on the space L_u is defined, in the spirit of Remark 3.3, by

$$[y,a]_s \cdot g := [yg, a]_{sg}.$$

This action clearly makes $\pi_u : L_u \to \mathcal{S}_p(G)$ into a G-map. In view of the above, G acts linearly on the fibers of π_u and thus makes L_u into a G-equivariant complex line bundle over $\mathcal{S}_p(G)$. We can also observe that the collection of local trivializations $\alpha_s : \mathbb{1}_{Ys} \xrightarrow{\sim} (L_u)_{|Ys}$ given in (3.4) is "*G-coherent*" ([1]) by which we mean that for all $s, g \in G$ we have

$$g_*(\alpha_s) = \alpha_{sg} \tag{3.6}$$

as isomorphisms $\mathbb{1}_{Ysg} \overset{\sim}{\to} (L_u)_{|Ysg}$. This fact results directly from the definitions, see (2.9) and (3.4). Combining this with (3.5) we note for later use the formula:

$$g_*(\alpha_1) = \alpha_1 \cdot u(g) \qquad \text{over } Y \cap Yg \tag{3.7}$$

as isomorphisms $\mathbb{1}_{Y\cap Yg} \overset{\sim}{\to} (L_u)_{|Y\cap Yg}$, for all $g \in G$ such that $P \cap P^g \neq 1$.

3.8. **Proposition.** *For any two weak P-homomorphisms $u, v \in A_{\mathbb{C}}(G,P)$ we have a G-equivariant isomorphism $L_{uv} \simeq L_u \otimes L_v$ of complex line bundles over $\mathcal{S}_p(G)$.*

Proof. Note that the trivializations (3.4) of L_u are performed on the closed cover of $\mathcal{S}_p(G)$ given by $(Ys)_{s\in G}$, which is independent of u. So, it is the same cover for L_u, L_v and L_{uv}. The statement then follows from the observation that the following obvious isomorphisms over Ys (where we temporarily decorate the three morphisms α as $\alpha^{(u)}$, $\alpha^{(v)}$ and $\alpha^{(uv)}$ to distinguish the respective line bundles)

$$(L_u \otimes L_v)_{|Ys} \cong (L_u)_{|Ys} \otimes (L_v)_{|Ys} \xleftarrow[\simeq]{\alpha_s^{(u)} \otimes \alpha_s^{(v)}} \mathbb{1}_{Ys} \otimes \mathbb{1}_{Ys} \cong \mathbb{1}_{Ys} \xrightarrow{\alpha_s^{(uv)}} (L_{uv})_{|Ys}$$

[1] We do not say "G-equivariant" to avoid confusion.

patch together into a G-equivariant isomorphism $L_u \otimes L_v \overset{\sim}{\to} L_{uv}$ on $\mathcal{S}_p(G)$. Verification of this patching is immediate from (3.5) and the following agreement:

$$\begin{array}{ccc} \mathbb{1}_{Ys\cap Yt} \otimes \mathbb{1}_{Ys\cap Yt} & \cong & \mathbb{1}_{Ys\cap Yt} \\ \downarrow {\scriptstyle (\cdot u(st^{-1}))\otimes(\cdot v(st^{-1}))} & & \downarrow {\scriptstyle \cdot uv(st^{-1})} \\ \mathbb{1}_{Ys\cap Yt} \otimes \mathbb{1}_{Ys\cap Yt} & \cong & \mathbb{1}_{Ys\cap Yt} \end{array}$$

on the trivial bundle. Finally, the map $L_u \otimes L_v \overset{\sim}{\to} L_{uv}$ is G-equivariant because each $\{\alpha_s^{(\dots)}\}_{s\in G}$ is a G-coherent collection of maps, as we saw in (3.6). □

3.9. *Notation.* As in the Introduction, we denote by $\mathbb{L} : A_{\mathbb{C}}(G,P) \longrightarrow \mathrm{Pic}^G(\mathcal{S}_p(G))$ the homomorphism $u \mapsto [L_u]$ defined by Construction 3.1 and Proposition 3.8.

This homomorphism is easily seen to be natural in the following sense:

3.10. **Proposition.** *Let $G' \leq G$ be a subgroup containing P and consider the G'-subspace $\mathcal{S}_p(G') \subseteq \mathcal{S}_p(G)$. Then the following diagram*

$$\begin{array}{ccc} A_{\mathbb{C}}(G,P) & \xrightarrow{\mathbb{L}} & \mathrm{Pic}^G(\mathcal{S}_p(G)) \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ A_{\mathbb{C}}(G',P) & \xrightarrow{\mathbb{L}} & \mathrm{Pic}^{G'}(\mathcal{S}_p(G')) \end{array}$$

is commutative. □

3.11. *Example.* Let $u : G \to \mathbb{C}^*$ be a group homomorphism, *i.e.* a one-dimensional representation. Assume that u is trivial on P. One associates to u a weak P-homomorphism $\tilde{u} \in A_{\mathbb{C}}(G,P)$ by forcing (WH 2), *i.e.* by setting for every $g \in G$

$$\tilde{u}(g) := \begin{cases} u(g) & \text{if } P \cap P^g \neq 1 \\ 1 & \text{if } P \cap P^g = 1. \end{cases} \tag{3.12}$$

Then $L_{\tilde{u}}$ is isomorphic to the "constant" line bundle (in the sense of [Seg68]), that is, the line bundle $\mathbb{1}_u := \mathcal{S}_p(G) \times \mathbb{C}$ with action $(y,a)\cdot g = (yg, au(g))$. Indeed, inspired by Remark 3.3, one easily guesses the G-equivariant isomorphism $L_{\tilde{u}} \overset{\sim}{\to} \mathbb{1}_u$ by sending the class $[y,a]_s$ in $L_{\tilde{u}}$ (see Construction 3.1) to the point $(y, a\cdot u(s))$ in $\mathcal{S}_p(G) \times \mathbb{C} = \mathbb{1}_u$. Verifications are left to the reader.

The modification (3.12) of u into a weak homomorphism $\tilde{u}$ is irrelevant for the construction of $L_{\tilde{u}}$ since (3.2) only uses values $\tilde{u}(g)$ over the subset $Y \cap Yg$. Indeed, either $P \cap P^g = 1$ and this subset is empty, or $P \cap P^g \neq 1$ and $\tilde{u}(g) = u(g)$ anyway. Furthermore, the homomorphism $u \mapsto \tilde{u}$ is often injective, even after (post-) composition with $\mathbb{L}$. We do not use the latter but state it for peace of mind:

3.13. **Proposition.** *Suppose that $\mathcal{S}_p(G)$ is connected. Let $u : G \to \mathbb{C}^*$ be a group homomorphism which is trivial on P and such that the G-equivariant line bundle $\mathbb{1}_u \simeq \mathbb{L}(\tilde{u})$ is G-equivariantly trivial on $\mathcal{S}_p(G)$ (for instance if $\tilde{u} = 1$). Then $u = 1$.*

Proof. A G-equivariant isomorphism $\mathbb{1} \overset{\sim}{\to} \mathbb{1}_u$ is given by multiplication by a map $f : \mathcal{S}_p(G) \to \mathbb{C}^*$ such that $f(xg) = f(x)\cdot u(g)$ for all $g \in G$ and $x \in \mathcal{S}_p(G)$. Choose an integer $m \geq 1$ such that $u(g)^m = 1$. Then $f^m : \mathcal{S}_p(G) \to \mathbb{C}^*$ is a G-map. By Corollary 2.13, this f^m admits an m^{th} root in $\mathrm{Cont}_G(\mathcal{S}_p(G), \mathbb{C}^*)$, *i.e.* there exists a G-map $\hat{f} : \mathcal{S}_p(G) \to \mathbb{C}^*$ such that $\hat{f}^m = f^m$. Since $\mathcal{S}_p(G)$ is assumed connected, we have $\hat{f} = f \cdot \rho$ for some constant $\rho \in \mu_m(\mathbb{C})$; see Notation 2.1. Then f is also a G-map and the above relation $f(xg) = f(x) \cdot u(g)$ forces $u(g) = 1$ for all $g \in G$. $\square$

Assuming $\mathcal{S}_p(G)$ connected is a mild condition. According to [Qui78, Prop. 5.2], if $\mathcal{S}_p(G)$ is disconnected then the stabilizer $H \leq G$ of a component is a strongly p-embedded subgroup, and our discussion can be safely reduced from G to H.

4. The results

We now prove our main result, from which we will deduce Theorem 1.1 stated in the Introduction. Recall from Notation 3.9 the homomorphism $\mathbb{L} : A_{\mathbb{C}}(G,P) \to \mathrm{Pic}^G(\mathcal{S}_p(G))$, $u \mapsto [L_u]$, from the group of complex-valued weak P-homomorphisms (Def. 1.2) to the G-equivariant Picard group (Rem. 2.5) of the Brown complex $\mathcal{S}_p(G)$.

4.1. **Theorem.** *The homomorphism $\mathbb{L} : A_{\mathbb{C}}(G,P) \to \mathrm{Pic}^G(\mathcal{S}_p(G))$ is injective on torsion subgroups (denoted* Tors*) and its image is the kernel of restriction to one-dimensional representations of P, see* (2.6)*. In other words, the following sequence*

$$0 \longrightarrow \mathrm{Tors}\, A_{\mathbb{C}}(G,P) \xrightarrow{\ \mathbb{L}\ } \mathrm{Tors}\,\mathrm{Pic}^G(\mathcal{S}_p(G)) \xrightarrow{\mathrm{Res}^G_P} \mathrm{Hom}_{\mathrm{gps}}(P,\mathbb{C}^*) \tag{4.2}$$

is exact. Consequently, for every integer $m \geq 1$ prime to p, our $\mathbb{L}$ restricts to an isomorphism on the m-torsion subgroups ([2])

$$\mathbb{L} : \mathrm{Tors}_m\, A_{\mathbb{C}}(G,P) \overset{\sim}{\to} \mathrm{Tors}_m\,\mathrm{Pic}^G(\mathcal{S}_p(G)).$$

Proof. The proof will occupy the next couple of pages. First note that by naturality of $\mathbb{L}$ (Prop. 3.10 applied to $G' = P$), the following square commutes:

$$\begin{array}{ccc} A_{\mathbb{C}}(G,P) & \xrightarrow{\ \mathbb{L}\ } & \mathrm{Pic}^G(\mathcal{S}_p(G)) \\ \Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\ 0 = A_{\mathbb{C}}(P,P) & \xrightarrow{\ \mathbb{L}\ } & \mathrm{Pic}^P(\mathcal{S}_p(P)) \cong \mathrm{Hom}_{\mathrm{gps}}(P,\mathbb{C}^*)\,. \end{array}$$

This proves that $\mathrm{Res}^G_P \circ \mathbb{L}$ is trivial (even outside torsion).

We now prove injectivity of $\mathbb{L}$ on the torsion of $A_{\mathbb{C}}(G,P)$. Let $u \in A_{\mathbb{C}}(G,P)$ be an element of m-torsion for some $m \geq 1$, meaning that $u(g)^m = 1$ for all $g \in G$. Suppose that we have a G-equivariant trivialization $\psi : \mathbb{1}_{\mathcal{S}_p(G)} \overset{\sim}{\to} L_u$ of the line bundle $\mathbb{L}(u) = L_u$ (see Constr. 3.1). Comparing the restriction $\psi_{|_Y}$ to the trivialization $\alpha_1 : \mathbb{1}_Y \overset{\sim}{\to} (L_u)_{|_Y}$ given in (3.4), we find a P-map $\delta : Y \to \mathbb{C}^*$ with

$$\psi_{|_Y} = \alpha_1 \cdot \delta$$

as isomorphisms $\mathbb{1}_Y \overset{\sim}{\to} (L_u)_{|_Y}$. Combining the G-equivariance of ψ with the relation $g_*(\alpha_1) = \alpha_1 \cdot u(g)$ on $Y \cap Yg$ from (3.7), we see that for every $g \in G$ such that $P \cap P^g \neq 1$, we have for every $y \in Y \cap Yg$

$$u(g) = \frac{\delta(y)}{g_*\delta(y)} = \frac{\delta(y)}{\delta(yg^{-1})}. \tag{4.3}$$

As the left-hand side belongs to $\mu_m(\mathbb{C})$, we deduce that δ^m and $g_*(\delta^m)$ agree on the intersection $Y \cap Yg$. Consequently the family of functions $(g_*(\delta^m))_{g\in G}$ patch together into a G-map $f : \mathcal{S}_p(G) \to \mathbb{C}^*$ by setting $f(x) = \delta(xg^{-1})^m$ whenever $x \in$

[2] By "m-torsion" we mean exactly the annihilator of m itself, not of powers of m.

Yg. By Corollary 2.13, f admits an m^{th} root, *i.e.* there exists a G-map $f^{1/m}: \mathcal{S}_p(G) \to \mathbb{C}^*$ such that $(f^{1/m})^m = f$. On Y, the two roots $f^{1/m}$ and δ of the same map f must differ by an m^{th} root $\rho \in \mu_m(\mathbb{C})$ which must be constant since Y is connected, say $\delta = \rho \cdot f^{1/m}$. But then for every $g \in G$ such that $P \cap P^g \neq 1$ and for any $y \in Y \cap Yg \neq \varnothing$ (for which $yg^{-1} \in Y$ too), relation (4.3) becomes

$$u(g) = \frac{\delta(y)}{\delta(yg^{-1})} = \frac{\rho \cdot f^{1/m}(y)}{\rho \cdot f^{1/m}(yg^{-1})} = 1$$

by G-equivariance of $f^{1/m}$. In the other case where $P \cap P^g = 1$, we have $u(g) = 1$ by (WH 2). In short, $u = 1$ is trivial. This proof uses the contractibility of $\mathcal{S}_p(G)/G$, since Corollary 2.13 relies on Symonds [Sym98].

We now prove exactness of (4.2) in the middle via another construction.

4.4. **Construction.** Let L be a G-equivariant complex line bundle on $\mathcal{S}_p(G)$, which is torsion and such that $\mathrm{Res}^G_P(L) = 1$, *i.e.* L restricts to the trivial P-bundle on $\mathcal{S}_p(P)$. *Choose* for some $m \geq 1$ a G-equivariant isomorphism

$$\omega : \mathbb{1}_{\mathcal{S}_p(G)} \xrightarrow{\sim} L^{\otimes m}$$

over $\mathcal{S}_p(G)$ and *choose* a P-equivariant isomorphism over $Y = \mathcal{S}_p(P)$

$$\beta : \mathbb{1}_Y \xrightarrow{\sim} L_{|_Y}$$

between the trivial bundle $\mathbb{1}_Y = Y \times \mathbb{C}$ and the restriction of L to Y. The P-equivariance of β means that, for every $h \in P$, we have

$$h_*(\beta) = \beta \tag{4.5}$$

as isomorphisms $\mathbb{1}_Y \xrightarrow{\sim} L_{|_Y}$. There is a choice in the isomorphism β, and we can replace β by $\beta \cdot \delta$ for any P-map $\delta : Y \to \mathbb{C}^*$. We shall use this flexibility shortly.

Observe that $\beta^{\otimes m}$ yields another trivialization of $L^{\otimes m}$ on Y, that we can compare to the initial ω, restricted to Y. It follows that we have $\omega_{|_Y} = \beta^{\otimes m} \cdot \epsilon$ for some P-map $\epsilon : Y \to \mathbb{C}^*$. Since the space Y is P-contractible, Corollary 2.12 produces an m^{th}-root of ϵ, say $\epsilon^{1/m} \in \mathrm{Cont}_P(Y, \mathbb{C}^*)$ with $(\epsilon^{1/m})^m = \epsilon$. Using this unit to replace β by $\beta \cdot \epsilon^{1/m}$, we can and shall assume that $\beta : \mathbb{1}_Y \xrightarrow{\sim} L_{|_Y}$ moreover satisfies

$$\beta^{\otimes m} = \omega_{|_Y}. \tag{4.6}$$

Then, for each $g \in G$, consider as before the translate $Yg = \mathcal{S}_p(P^g) \subseteq \mathcal{S}_p(G)$ and transport β into an isomorphism $g_*(\beta) : \mathbb{1}_{Yg} \xrightarrow{\sim} L_{|_{Yg}}$; see (2.9). If the isomorphisms β and $g_*(\beta)$ were to agree on the intersection of their domains of definition $Y \cap Yg$ for all $g \in G$, the collection of isomorphisms $(g_*(\beta))_{g \in G}$ would patch together into a global isomorphism $\mathbb{1}_{\mathcal{S}_p(G)} \xrightarrow{\sim} L$, automatically G-equivariant by construction. Since this cannot happen for nontrivial L, there is an obstruction, and this happens to be a weak P-homomorphism. Indeed, for every $g \in G$ such that $P \cap P^g \neq 1$, define what is *a priori* a function $u_L(g) \in \mathrm{Cont}(Y \cap Yg, \mathbb{C}^*)$ by

$$g_*(\beta) = \beta \cdot u_L(g) \qquad \text{over } Y \cap Yg \tag{4.7}$$

i.e. by the commutativity of the following diagram of line bundles on $Y \cap Yg$:

$$\begin{tikzcd}
\mathbb{1}_{Y\cap Yg} \arrow[r, "(g_*(\beta))_{|Y\cap Yg}", "\simeq"'] \arrow[d, "\cdot u_L(g):="', "\simeq"] & (L_{|Yg})_{|Y\cap Yg} = L_{|Y\cap Yg} \arrow[d, equal] \\
\mathbb{1}_{Y\cap Yg} \arrow[r, "\beta_{|Y\cap Yg}", "\simeq"'] & (L_{|Y})_{|Y\cap Yg} = L_{|Y\cap Yg}\,.
\end{tikzcd} \tag{4.8}$$

There is no choice at this step. By convention, we set

$$u_L(g) = 1 \qquad \text{if} \qquad P \cap P^g = 1. \tag{4.9}$$

In the case $P \cap P^g \neq 1$, we are going to prove that $u_L(g) : Y \cap Yg \to \mathbb{C}^*$ is a constant function. Taking (4.8) to the m^{th} tensor power, replacing both instances of $\beta^{\otimes m}$ by ω thanks to (4.6) and using that ω is G-equivariant, we deduce that $(u_L(g))^m = 1$ on $Y \cap Yg$. Since this space is non-empty and connected (even contractible), this implies that the function $u_L(g)$ is actually constant, with value equal to some complex m^{th} root of unity $u_L(g) \in \mu_m(\mathbb{C})$. In other words, the function

$$u_L : G \to \mu_m(\mathbb{C})\ , \qquad g \mapsto u_L(g)$$

is a candidate to be a complex-valued weak P-homomorphism. It satisfies (WH 1) by P-equivariance of β, see (4.5) and (4.8) for $g = h \in P$; and u_L satisfies (WH 2) by definition (4.9). To verify the last property (WH 3), consider $g_1, g_2 \in G$ such that $P \cap P^{g_1} \cap P^{g_2 g_1} \neq 1$, *i.e.* such that the subset $Z := Y \cap Yg_1 \cap Yg_2g_1$ is non-empty. Then juxtaposing the defining diagram (4.8) for $u_L(g_1)$ and the one for $u_L(g_2)$ transported by $(g_1)_*$, both suitably restricted to this triple intersection Z, we obtain the following commutative diagram over Z:

$$\begin{tikzcd}
\mathbb{1}_Z \arrow[r, "(g_{1*}g_{2*}(\beta))_{|Z}", "\simeq"'] \arrow[d, "g_{1*}(\cdot u_L(g_2)) = \cdot u_L(g_2)"', "\simeq"] & L_{|Z} \arrow[d, equal] \\
\mathbb{1}_Z \arrow[r, "(g_{1*}\beta)_{|Z}", "\simeq"'] \arrow[d, "\cdot u_L(g_1)"', "\simeq"] & L_{|Z} \arrow[d, equal] \\
\mathbb{1}_Z \arrow[r, "\beta_{|Z}", "\simeq"'] & L_{|Z}\,.
\end{tikzcd} \tag{4.10}$$

We used at the top left that $g_{1*}(-)$ is $\mathbb{C}$-linear. Using now that $g_{1*}g_{2*} = (g_2g_1)_*$, the left-hand vertical composite satisfies the commutativity expected of $u_L(g_2g_1)$, *i.e.* fits in place of $u_L(g_2g_1)$ in (4.8) for $g = g_2g_1$, after restriction of the latter to Z. This is where we use that $Z \neq \varnothing$ to deduce that $u_L(g_2g_1) = u_L(g_2) \cdot u_L(g_1)$.

It is interesting to see the parallel of these arguments with those of [Bal13], where the non-emptiness of Z is replaced by the non-vanishing of a suitable stable category. Both properties are equivalent, namely they both are avatars of the fact that the Sylow P and its conjugates P^{g_1} and $P^{g_2g_1}$ intersect non-trivially.

At this stage, we have associated a weak P-homomorphism $u_L \in \mathrm{Tors}_m\, A_{\mathbb{C}}(G,P)$ to an m-torsion G-equivariant line bundle L on $\mathcal{S}_p(G)$ and choices of isomorphisms $\omega : \mathbb{1}_{\mathcal{S}_p(G)} \overset{\sim}{\to} L^{\otimes m}$ and $\beta : \mathbb{1}_Y \overset{\sim}{\to} L_{|Y}$ satisfying (4.6). We now claim that $\mathbb{L}(u_L) \simeq L$. For this, recall the line bundle L_{u_L} of Construction 3.1, which describes $\mathbb{L}(u_L)$. It comes with an isomorphism $\alpha_1 : \mathbb{1}_Y \overset{\sim}{\to} (L_{u_L})_{|Y}$ satisfying

$$g_*(\alpha_1) = \alpha_1 \cdot u_L(g) \qquad \text{over } Y \cap Yg$$

by (3.7). Comparing this formula to the similar one for β in (4.7), we see that the following isomorphism $\varphi := \beta \circ \alpha_1^{-1}$ over Y

$$\varphi \,:\, (L_{u_L})_{|Y} \xrightarrow[\simeq]{\alpha_1^{-1}} \mathbb{1}_Y \xrightarrow[\simeq]{\beta} L_{|Y}$$

satisfies $g_*(\varphi) = \varphi$ on $Y \cap Yg$ for all $g \in G$. Therefore, the $(g_*\varphi)_{g\in G}$ patch together into a morphism $\varphi : L_{u_L} \to L$ which is G-equivariant and an isomorphism by construction. This finishes the proof of the exactness of the sequence (4.2).

It is immediate that $\mathbb{L}$ restricts to an isomorphism on prime-to-p torsion since $\mathrm{Hom}_{\mathrm{gps}}(P, \mathbb{C}^*)$ is p^r-torsion, where $|P| = p^r$, hence every $L \in \mathrm{Tors}_m \mathrm{Pic}^G(\mathcal{S}_p(G))$ with m prime to p maps to zero under Res^G_P.

This finishes the proof of Theorem 4.1. □

4.11. *Remark.* Construction 4.4 describes the inverse of $\mathbb{L}$ on prime-to-p torsion.

Let us now connect these results over $\mathbb{C}$ to positive characteristic objects. We recall some well-known facts, to facilitate cognition.

4.12. *Remark.* The group $\mathrm{T}_{\Bbbk}(G, P)$ is always finite. (Indeed, every endotrivial module in $\mathrm{T}_{\Bbbk}(G, P)$ is a direct summand of $\Bbbk(G/P)$ – an explicit projector depending on $u \in A_{\Bbbk}(G, P)$ is given in [Bal13]. By Krull-Schmidt it follows that $\mathrm{T}_{\Bbbk}(G, P)$ has at most $\dim_{\Bbbk}(\Bbbk(G/P)) = [G : P]$ elements.) Also, the order of $\mathrm{T}_{\Bbbk}(G, P)$ is prime to p; see [Bal13, Cor. 5.3]. For an algebraic closure $\bar{\Bbbk}$ of $\Bbbk$, one can easily identify the image of $\mathrm{T}_{\Bbbk}(G, P) \hookrightarrow \mathrm{T}_{\bar{\Bbbk}}(G, P)$; see [Bal13, Cor. 5.5].

In fact, the group $\mathrm{T}_{\Bbbk}(G, P)$ "stabilizes" once $\Bbbk$ *contains all roots of unity* by which we mean it contains all m^{th} roots of unity for all integers $m \geq 1$ prime to p. Here, "stabilization" means that $\mathrm{T}_{\Bbbk}(G, P) \to \mathrm{T}_{\Bbbk'}(G, P)$ is an isomorphism for every further extension $\Bbbk \to \Bbbk'$; see [Bal13, Cor. 5.5]. This condition is of course fulfilled if the field $\Bbbk = \bar{\Bbbk}$ is algebraically closed, or simply if $\Bbbk$ contains $\bar{\mathbb{F}}_p$, the algebraic closure of the prime field. Our Theorem 1.1 is another way of seeing why $\mathrm{T}_{\Bbbk}(G, P)$ stabilizes once $\Bbbk$ contains all roots of unity, by giving it a topological interpretation:

4.13. **Corollary.** *The prime-to-p torsion* $\mathrm{Tors}_{p'} \mathrm{Pic}^G(\mathcal{S}_p(G))$ *is a finite subgroup of* $\mathrm{Pic}^G(\mathcal{S}_p(G))$*. For any field* $\Bbbk$ *of characteristic p which contains all roots of unity (see Remark 4.12), we have an isomorphism as announced in Theorem 1.1*

$$\mathrm{T}_{\Bbbk}(G, P) \simeq \ \mathrm{Tors}_{p'} \mathrm{Pic}^G(\mathcal{S}_p(G))$$

where $\mathrm{Tors}_{p'}$ *denotes the prime-to-p torsion subgroup.*

Proof. Let $\Bbbk$ contain all roots of unity (or just the $[G\!:\!P]^{\text{th}}$-roots) and let e be the exponent of $\mathrm{T}_{\Bbbk}(G, P)$. Let $m \geq 1$ be an integer, prime to p and divisible by e.

By (1.3), the integer e is also the exponent of $A_{\Bbbk}(G, P) \simeq \mathrm{T}_{\Bbbk}(G, P)$ hence $u^m = 1$ for all $u \in A_{\Bbbk}(G, P)$. Thus every $u : G \to \Bbbk^*$ in $A_{\Bbbk}(G, P)$ takes values in $\mu_m(\Bbbk)$. In other words, we can identify the group of $\Bbbk$-*valued* weak P-homomorphisms $A_{\Bbbk}(G, P)$ with the set of functions $u : G \to \mu_m(\Bbbk)$ satisfying (WH 1-3).

Consider now inside the group $A_{\mathbb{C}}(G, P)$ of *complex-valued* weak P-homomorphisms, the subgroup $\mathrm{Tors}_m A_{\mathbb{C}}(G, P)$ of elements of order dividing m. Again, this is just the subset of those functions $u : G \to \mu_m(\mathbb{C})$ satisfying (WH 1-3).

Choose now an isomorphism $\mu_m(\Bbbk) \simeq \mathbb{Z}/m \simeq \mu_m(\mathbb{C})$. This uses that $\Bbbk$ contains all m^{th} roots of unity. Combining the above we obtain an isomorphism

$$A_{\Bbbk}(G, P) \simeq \mathrm{Tors}_m A_{\mathbb{C}}(G, P). \tag{4.14}$$

Since the left-hand side is independent of such m (prime to p and divisible by e), we get $\mathrm{Tors}_{p'} A_{\mathbb{C}}(G,P) = \mathrm{Tors}_e A_{\mathbb{C}}(G,P)$. Using now Theorem 4.1, it follows that $\mathrm{Tors}_{p'} \mathrm{Pic}^G(\mathcal{S}_p(G)) = \mathrm{Tors}_e \mathrm{Pic}^G(\mathcal{S}_p(G)) \simeq \mathrm{Tors}_e A_{\mathbb{C}}(G,P)$ via $\mathbb{L}$. The latter is itself isomorphic to $A_{\Bbbk}(G,P) \simeq \mathrm{T}_{\Bbbk}(G,P)$ by a last instance of (4.14) and (1.3). $\square$

4.15. *Remark.* The isomorphism of Corollary 4.13 is essentially induced by the canonical homomorphism $\mathbb{L} : A_{\mathbb{C}}(G,P) \to \mathrm{Pic}^G(\mathcal{S}_p(G))$ of Section 3, up to the choice of an identification between e^{th} roots of unity in $\Bbbk$ and e^{th} roots of unity in $\mathbb{C}$, for e the exponent of $\mathrm{T}_{\Bbbk}(G,P)$. Another choice of an isomorphism $\mu_e(\Bbbk) \simeq \mu_e(\mathbb{C})$ simply changes the isomorphism (4.14) by multiplication with some integer prime to e, a rather harmless operation which is of course invertible.

Combining the above with Example 3.11, we obtain:

4.16. **Corollary.** *The following properties of G and p are equivalent:*

(i) For $\Bbbk = \bar{\mathbb{F}}_p$ the group $\mathrm{T}_{\Bbbk}(G,P)$ consists only of one-dimensional representations $G \to \Bbbk^*$.

(i') For every field $\Bbbk$ containing all roots of unity, the group $\mathrm{T}_{\Bbbk}(G,P)$ consists only of one-dimensional representations $G \to \Bbbk^*$.

(ii) Every G-equivariant complex line bundle on $\mathcal{S}_p(G)$ which is torsion of order prime to p is constant, i.e. $\mathrm{Tors}_{p'} \mathrm{Pic}^G(*) \to \mathrm{Tors}_{p'} \mathrm{Pic}^G(\mathcal{S}_p(G))$ *is onto.* $\square$

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