TRIANGULATED CATEGORIES WITH SEVERAL TRIANGULATIONS

PAUL BALMER

Abstract. We give a simple algebraic example of a fixed additive category \( K \), with a fixed additive self-equivalence \( \Sigma : K \to K \) having arbitrarily many structures of triangulated categories with \( \Sigma \) as suspension.

Introduction. Triangulated categories were introduced in Verdier’s PhD thesis [3] and by Puppe (without Verdier’s key Octahedron Axiom about composition). A triangulated category is a suspended category (i.e. an additive category \( K \) with an additive auto-equivalence \( \Sigma : K \to K \) called the suspension) plus a collection \( \mathcal{T} \) of triangles which satisfies four well-known axioms, denoted (TR I)-(TR IV) in [3, Def. II.1.1.1, p. 93-94]. It is natural to wonder whether these axioms are intrinsical. Or: Can a given suspended category carry two different triangulations? Of course, given a triangulated category \( (K, \Sigma, \mathcal{T}) \) with triangulation \( \mathcal{T} \), we can define the negative triangulation \( \mathcal{T}^- \) as the class of those triangles \( (u, v, w) \) such that \((-u, -v, -w) \in \mathcal{T} \). Those two triangulations \( \mathcal{T} \) and \( \mathcal{T}^- \) are different in general, already for \( K = K^b(\mathbb{Z}) \) the category of bounded complexes of abelian groups, up to homotopy. Then, we could ask:

Can a suspended category \( (K, \Sigma) \) admit more than two triangulations?

(That is: one triangulation and its negative.) Strangely enough, this question seems to remain unclear, even for a few experts of the subject, see for instance [2, Problem 3.4 and Def. 3.2]. The answer to this question is indeed “yes”, and first in a trivial way: Let \( (K, \Sigma, \mathcal{T}) \) be a triangulated category such that \( \mathcal{T} \) and \( \mathcal{T}^- \) are different. Choose an integer \( n \geq 1 \). Consider the additive category \( K^n = K \times \cdots \times K \) with the obvious suspension. Then \( K^n \) has at least \( 2^n \) different triangulations compatible with its suspension. Well, this certainly sounds like cheating, because we basically only used \( \mathcal{T} \) and \( \mathcal{T}^- \). So, we would like to build examples, say, with an indecomposable category \( K \). In fact, it is possible to deduce from the results of Sections 16 and 17 of Heller [1] that such an example is given by \( K \) the usual topological stable homotopy category, although a picky reader might object that Heller does not consider the Octahedron Axiom in loc. cit. In this short note, we give a simple algebraic example (see Theorem 7).

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Definition 1. Let \( (K, \Sigma) \) be a suspended category. A global endomorphism \( \alpha \) of \( (K, \Sigma) \) will be an endomorphism of the identity functor \( \text{Id} : K \to K \) which commutes with \( \Sigma \). In other words, it consists of a collection of endomorphisms \( \alpha_A : A \to A \), for all objects \( A \in K \), such that for any morphism \( f : A \to B \) in \( K \) one has \( \alpha_B \circ f = f \circ \alpha_A \), and such that \( \alpha_{\Sigma(A)} = \Sigma(\alpha_A) \) for any \( A \in K \). A global automorphism will be an invertible global endomorphism. A global endomorphism \( \alpha \) is pointwise nilpotent if for any \( A \in K \), there is an \( n \in \mathbb{N} \) such that \((\alpha_A)^n = 0\).

Example 2. Let \( R \) be a commutative ring and let \( b \in R \). Then multiplication by \( b \) gives a global endomorphism \( \lambda_b \) of \( K^b(R) \). It is a global automorphism when \( b \) is a unit.

Definition 3. Let \( (K, \Sigma, \mathcal{T}) \) be a triangulated category and let \( \alpha \) be a global automorphism of \( (K, \Sigma) \). Define the class \( \mathcal{T}_\alpha \) as the collection of those triangles \( (u, v, w) \), like in (\( \Delta \)) above, such that the twisted triangle \( (u \cdot \alpha_A, v, w) \) belongs to \( \mathcal{T} \). This condition is equivalent to any of the following: \((\alpha_B \cdot u, v, w) \in \mathcal{T}, \ (u, v \cdot \alpha_B, w) \in \mathcal{T}, \ (u, \alpha_C \cdot v, w) \in \mathcal{T} \), and so on: permuting \( \alpha \) with the morphisms, and moving \( \alpha \) around; this flexibility follows from Axiom (TR I) and Def. 1.

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Proposition 4. Let \((K, \Sigma, T)\) and \(\alpha\) be as in Definition 3. Then \((K, \Sigma, T_{\alpha})\) is a triangulated category.

Proof. The proof is straightforward. The Composition Axiom (TR IV), for instance, can be checked by contemplating the diagram on the left (or the reader’s favorite picture instead). The arrows with a small dot are of degree 1. Start with a composition \(w = v \circ u\).

Lemma 5. Let \(R\) be a commutative ring and \(b \in R^\times\) a unit. Assume the existence of \(r \in R\) such that:
(1) the element \(r\) is not a zero divisor and (2) the element \(r\) does not divide \(1 - b\).

Consider the category \(K = K^b(R)\) with its usual triangulation \(T\). Consider the global automorphism \(\lambda_b\) of \(K\) (see Example 2). Then the triangulations \(T\) and \(T_{\lambda_b}\) are different.

Proof. Ab absurdum, assume that \(T_{\lambda_b} = T\). In the category \(K\), consider the morphism \(R \rightarrow R\) given by multiplication by \(r\) as a morphism of complexes concentrated in degree 0. Let \(C(r)\) be its cone with the usual morphisms \(i : R \rightarrow C(r)\) and \(p : C(r) \rightarrow \Sigma(R)\). The triangle \((b \cdot r, i, p)\) is then exact. By (TR III), there must exist a morphism \(h : C(r) \rightarrow C(r)\) which makes the following diagram commute:

\[
\begin{array}{ccc}
R & \xrightarrow{r} & R \\
\downarrow & & \downarrow \\
C(r) & \xrightarrow{i} & C(r) \\
\downarrow & & \downarrow \\
C(r) & \xrightarrow{p} & \Sigma(R) \\
\end{array}
\]

The morphism \(h\) is characterized by two elements \(x, y \in R\) such that \(r \cdot x = y \cdot r\) which forces \(x = y\) by hypothesis (1). The commutativity (up to homotopy!) of the above diagram implies the existence of \(e, f \in R\) such that \(b = x + r \cdot e\) and \(1 = x + f \cdot r\). This gives \(1 - b \in rR\) which contradicts (2).

Example 6. Of course \(T_{-1d} = T^{-}\). The Lemma shows that \(T^{-} \neq T\) for \(K^b(\mathbb{Z})\) as claimed above.

Theorem 7. There exists a suspended category \((K, \Sigma)\) which carries infinitely many different triangulations. Moreover, there exists such a \((K, \Sigma)\) which cannot be decomposed as \((K_1, \Sigma_1) \times (K_2, \Sigma_2)\) with \(K_1\) and \(K_2\) non-zero.

Proof. Let \(S\) be a commutative domain with infinitely many units. Let \(R = S[X]\) be the polynomial ring with coefficients in \(S\). Consider \(r = X \in R\). It certainly satisfies conditions (1) and (2) of the above Lemma for any unit \(b \in S^\times\) except for \(b = 1\). Let us write \(T_b\) for \(T_{\lambda_b}\). It is clear that \((T_b)c = T_{b \cdot c}\) for any \(b, c \in S^\times\). Therefore \(T_b = T_c\) forces \(T = T_{-1c}\) and thus \(b^{-1}c = 1\) by the Lemma and the above comment. That is: all the triangulations \(T_b\) for \(b \in S^\times\) are distinct.

For the “moreover part”, assume \((K, \Sigma) = (K_1, \Sigma_1) \times (K_2, \Sigma_2)\) then the projection on the \(K_1\)-summand yields a global endomorphism \(\beta\) of \(K\), see Definition 1. At the object \(R \in K\), we have necessarily \(\beta_R = 0\) or \(1 - \beta_R = 0\), since \(\text{End}_K(R) \simeq R\) and since \(R\) is a domain. Let us say \(\beta_R = 0\) for instance. The object \(R\) generates \(K\) as a triangulated category, this forces \(\beta\) to be pointwise nilpotent (easy induction). But \(\beta = \beta^2\) is an idempotent, so we have \(\beta = 0\) and thus \(K_1 = 0\). Similarly if \(1 - \beta_R = 0\), then \(K_2 = 0\).

Problem 8. Is there a suspended category \((K, \Sigma)\) admitting two triangulations \(T\) and \(T'\) such that \(T' \neq T_{\alpha}\) for any global automorphism \(\alpha\) of \((K, \Sigma)\)?