SEPARABILITY AND TRIANGULATED CATEGORIES

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Abstract. We prove that the category of modules over a separable ring object in a tensor triangulated category admits a unique structure of triangulated category which is compatible with the original one. This applies in particular to étale algebras. More generally, we do this for exact separable monads.

Contents

Introduction

Given a ring spectrum in the topological stable homotopy category, or more generally a ring object in any tensor triangulated category, the modules over that ring do not form a triangulated category in any obvious way. This might be considered a serious drawback of triangulated categories. To circumvent that problem, one usually needs to descend to some model, consider the category of modules down there, and then take the homotopy category to expect producing some reasonable triangulated category. The aesthetical and technical costs of this complication are evident. Our purpose here is to prove that obstacles vanish when the ring object in question is separable (hence solving an old private conjecture of Giordano Favi) :

Main Theorem. *Let $\mathcal{C}$ be a tensor triangulated category and A a* separable *ring object in $\mathcal{C}$, meaning that the multiplication $\mu : A\otimes A \to A$ has a bimodule section. Then the category of left A-modules in $\mathcal{C}$ has a triangulation in which distinguished triangles are the ones whose underlying triangle of objects is distinguished in $\mathcal{C}$.*

Note that the notion of *A-module in $\mathcal{C}$* and that of *separability* are the standard ones, repeated in Definitions 2.4 and 3.1 respectively. What is new here is their harmonious interaction with the triangular structure. We are actually going to

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prove this theorem in greater generality, without tensor structure on $\mathcal{C}$, replacing the ring object A by any exact separable monad (Def. 3.5). By duality, our results extend to co-modules over co-rings, or co-monads; see Remark 5.19. But for this short introduction, let us stick to ordinary rings and modules.

The true interest of our Main Theorem is that it offers a new type of construction that can be performed on triangulated categories, without descending to models. There are actually very few such general constructions, beyond localization of course, which is arguably the most important one. Interestingly, we shall see in Example 6.3 that Bousfield localization is a special case of our construction.

There is a disclaimer to be made about the above result, which brings us to an important second theme of the paper. As stated, the theorem only holds for *pre*-triangulated categories, that is, without Verdier's octahedron axiom. The reason is that Verdier's axiom might be perfectible, as also indicated by the recent work of Matthias Künzer [17]. The natural improvement consists in requiring a morphism axiom for octahedra, analogous to the morphism axiom for triangles. These considerations extend to higher octahedra *à la* Beilinson-Bernstein-Deligne [5]. Let us postpone this somewhat technical discussion to Section 5 and simply say that there *is* a way to improve the axiomatic, due to Künzer [15, 16]; see alternatively Maltsiniotis [19] for a neat compact presentation. Comfortingly, this improved axiomatic is satisfied by the homotopy category of any stable model category, so there is no real restriction in terms of applications; see Remarks 5.12 and 5.14. With this improved axiomatic, our theorem holds true and actually extends to any higher order of triangulation, including the infinite one, as we shall see in Theorem 5.17.

The organization of the paper is the following : Sections 1, 2 and 3 recall standard material, with minor modifications for compatibility with the suspension. The work starts in Section 4, where we prove the weak version of our Main Theorem without the octahedron. In Section 5, we present the higher axioms and prove the full fledge version of the result. We finally provide examples in Section 6. For instance, Theorem 6.5 and Corollary 6.6 give us :

Theorem. *Let R be a commutative ring and let A be a flat and separable R-algebra (e.g. a commutative étale R-algebra). Then the derived category of A-modules* $\mathrm{D}(A\text{–Mod})$ *is triangular equivalent to the category of A-modules in* $\mathrm{D}(R\text{–Mod})$.

In terms of tensor triangular geometry [3], this opens the way to *étale morphisms of tensor triangulated categories*, extending the theory of [2] beyond the Zariski topology. This will be the subject of forthcoming work.

Finally, let us stress the fact that separability is important beyond algebra and algebraic geometry. For instance, Rognes [23] offers a thorough investigation of (commutative) separable and étale algebras in stable homotopy theory and provides many examples. The reader interested in this direction is also referred to the recent work of Baker-Richter [1] and Hess [11].

1. Pre-triangulated categories

1.1. *Definition.* A *suspended category* (or *triangulated category of first order*) is an additive category $\mathcal{C}$ with an auto-equivalence $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ that we call the *suspension*. For simplicity, we consider Σ as an isomorphism, $\Sigma^{-1}\Sigma = \mathrm{Id}_{\mathcal{C}} = \Sigma\Sigma^{-1}$, to avoid overloading the notation with natural isomorphisms. A *triangle* in $\mathcal{C}$ is a diagram $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$, often represented as

$$
\text{(1.2)} \qquad \begin{tikzcd} & c \arrow[dl, dashed, "h"'] & \\ a \arrow[rr, "f"'] & & b \arrow[ul, "g"'] \end{tikzcd} \qquad \text{or even} \qquad \begin{tikzcd} a \arrow[r, "f"] & b \arrow[d, "g"] & \\ & c \arrow[r, "h"] & \Sigma(a) \end{tikzcd} \qquad \text{if space permits.}
$$

The morphism $f : a \to b$ is called the *base* of the triangle. Morphisms of triangles are the obvious morphisms of (periodic) diagrams, see (1.4) below. The broken arrow $c \dashrightarrow a$ indicates a morphism of degree one from a to c, that is, $c \to \Sigma\, a$.

1.3. *Definition.* A *pre-triangulated category* (or *triangulated category of second order*) is a suspended category $(\mathcal{C}, \Sigma)$ as above, together with a collection of *distinguished triangles* (a. k. a. *exact* triangles) subject to the following axioms :

(TC 2.1) *Bookkeeping Axioms* :

(TC 2.1.a) Every triangle isomorphic to a distinguished triangle is distinguished.

(TC 2.1.b) For every object a in $\mathcal{C}$, the two triangles $0 \to a \xrightarrow{1} a \to 0$ and $a \xrightarrow{1} a \to 0 \to \Sigma a$ are distinguished.

(TC 2.1.c) A triangle $\Delta = \big(a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a \big)$ is distinguished if and only if

$$\sigma(\Delta) := \big(\Sigma a \xrightarrow{\Sigma f} \Sigma b \xrightarrow{\Sigma g} \Sigma c \xrightarrow{-\Sigma h} \Sigma^2 a \big)$$

is distinguished; and Δ is distinguished if and only if

$$\tau(\Delta) := \big(c \xrightarrow{h} \Sigma a \xrightarrow{\Sigma f} \Sigma b \xrightarrow{\Sigma g} \Sigma c \big)$$

is distinguished.

(TC 2.2) *Existence Axiom* : Every morphism $f : a \to b$ is the base of some distinguished triangle.

(TC 2.3) *Morphism Axiom* : For every pair of distinguished triangles, every morphism on their bases, as in the following left-hand commutative square :

$$
\text{(1.4)} \qquad \begin{tikzcd} \Delta \arrow[d, phantom] & = & a \arrow[r, "f"] \arrow[d, "\alpha"'] & b \arrow[r, "g"] \arrow[d, "\beta"'] & c \arrow[r, "h"] \arrow[d, dotted, "\exists\,\gamma"'] & \Sigma(a) \arrow[d, "\Sigma\alpha"] \\ \Delta' & = & a' \arrow[r, "f'"'] & b' \arrow[r, "g'"'] & c' \arrow[r, "h'"'] & \Sigma(a') \end{tikzcd},
$$

extends to a morphism of triangles $(\alpha, \beta, \gamma) : \Delta \longrightarrow \Delta'$, meaning of course that all three squares above commute.

This notion of pre-triangulated category is the same as in Neeman [21, Def. 1.1.2], except for the numbering of the axioms, which is not important. An additive functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ between pre-triangulated categories is called *exact* if it commutes with the suspensions and preserves distinguished triangles.

1.5. *Remarks.* It is intuitively clear that the "meat" of Definition 1.3 is contained in axioms (TC 2.2) and (TC 2.3), which assert the *existence* of triangles and morphisms thereof. The other axioms are only shuffling existing information around.

We have introduced some redundancy, in the bookkeeping axioms for instance, to make the analogy with the higher axiomatic of Section 5 more transparent.

A *triangulated category* in the sense of Verdier [24] consists of a pre-triangulated category which satisfies moreover the octahedron axiom. We shall return to this point in Remark 5.8.

1.6. **Lemma.** *Let $\mathcal{C}$ be a suspended category with a collection of distinguished triangles satisfying the bookkeeping axioms (TC 2.1) and the morphism axiom (TC 2.3). In other words, $\mathcal{C}$ is almost a pre-triangulated category, except for axiom (TC 2.2).*

Let $\Delta = \big(a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a \big)$ be a distinguished triangle. Then:

(a) f is a weak kernel of g and h is a weak cokernel of g.

(b) If $k : c \to c$ is such that $(0, 0, k)$ is an endomorphism of Δ, then $k^2 = 0$.

(c) If $d = (p, q, s)$ is an endomorphism of the triangle Δ which is an idempotent on the base, that is, $p^2 = p$ and $q^2 = q$, then $e := 3d^2 - 2d^3 = (p, q, 3s^2 - 2s^3)$ is an idempotent endomorphism, $e^2 = e$, of Δ.

(d) Let $p = p^2 : a \to a$ and $q = q^2 : b \to b$ be idempotents such that $fp = qf$:

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"'] & b \arrow[r, "g"] \arrow[d, "q"'] & c \arrow[r, "h"] \arrow[d, dotted, "\exists\, r"'] & \Sigma(a) \arrow[d, "\Sigma p"] \\
a \arrow[r, "f"'] & b \arrow[r, "g"'] & c \arrow[r, "h"'] & \Sigma(a)\,.
\end{tikzcd}
\tag{1.7}
$$

Then there exists an idempotent $r = r^2 : c \to c$ such that (p, q, r) is a morphism.

Proof. This is standard. For instance, the proof of (a) in [21, 1.1.3 and 1.1.10] applies *verbatim* and doesn't use the existence of triangles – our missing (TC 2.2). Point (b) follows from (a); indeed $kg = 0$ implies that $k = \bar{k}\, h$ for some $\bar{k}$ and then $k^2 = \bar{k}hk = 0$ since $hk = 0$. Then, (c) follows from (b) as in [4, 1.14]: $d^2 - d = (0, 0, s^2 - s)$ is also an endomorphism of Δ; hence $k := s^2 - s$ squares to zero by (b) and commutes with s; then, $s + k - 2sk = 3s^2 - 2s^3$ is an idempotent by direct computation. (This is the trick of lifting idempotents modulo nilpotence.) Part (d) follows easily from the morphism axiom (TC 2.3) and part (c). □

1.8. *Definition.* Recall that an additive category $\mathcal{C}$ is *idempotent-complete* if every idempotent morphism $e = e^2 : x \to x$ *splits*, that is, there is a decomposition $x = \mathrm{im}(e) \oplus \mathrm{im}(1-e)$, under which e becomes $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Any additive category admits an *idempotent completion* $\iota : \mathcal{C} \longrightarrow \mathcal{C}^\natural$. We proved in [4] that for $\mathcal{C}$ (pre-) triangulated, $\mathcal{C}^\natural$ inherits a unique structure of (pre-) triangulated category such that ι is exact.

1.9. *Remark.* Let $\mathcal{C}$ be an idempotent-complete additive category and I be a small category. It is well-known that the additive category $\mathrm{Fun}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$ and natural transformations (I-shaped diagrams in $\mathcal{C}$) is idempotent-complete as well. Indeed, let $F \in \mathrm{Fun}(I, \mathcal{C})$ and $e = e^2 : F \to F$ an idempotent natural transformation. Denote by $e' = 1 - e$ the idempotent complement. Since $\mathcal{C}$ is idempotent-complete, we split every object $F(i) = \mathrm{im}(e_i) \oplus \mathrm{im}(e'_i)$ in $\mathcal{C}$. For every $\alpha : i \to j$ the morphisms $F(\alpha) : F(i) \to F(j)$ commutes with the idempotents:

$e_jF(\alpha) = F(\alpha)e_i$ which forces $F(\alpha)$ to be diagonal: $F(\alpha) = \left(\begin{smallmatrix} e_jF(\alpha)e_i & 0 \\ 0 & e'_jF(\alpha)e'_i \end{smallmatrix}\right)$. Then $F = G \oplus G'$ where $G(i) = \mathrm{im}(e_i)$ for every $i \in I$ and $G(\alpha) = e_jF(\alpha)e_i$ for every $\alpha : i \to j$, and similarly for G' with e' instead of e.

1.10. **Proposition.** *Let $\mathcal{C}$ be an idempotent-complete suspended category and let $e = (p, q, r)$ be an idempotent endomorphism of a triangle $\Delta = \big(a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a \big)$; see (1.7). Then the triangle Δ is the direct sum of two triangles, as follows:*

$$\begin{matrix} \mathrm{im}(p) & & \mathrm{im}(q) & & \mathrm{im}(r) & & \Sigma\,\mathrm{im}(p) \\ \oplus & \xrightarrow{\left(\begin{smallmatrix} qfp & 0 \\ 0 & q'fp' \end{smallmatrix}\right)} & \oplus & \xrightarrow{\left(\begin{smallmatrix} rgq & 0 \\ 0 & r'gq' \end{smallmatrix}\right)} & \oplus & \xrightarrow{\left(\begin{smallmatrix} \Sigma(p)hr & 0 \\ 0 & \Sigma(p')hr' \end{smallmatrix}\right)} & \oplus \\ \mathrm{im}(p') & & \mathrm{im}(q') & & \mathrm{im}(r') & & \Sigma\,\mathrm{im}(p') \end{matrix}$$

where $(p', q', r') = (1-p\,,\,1-q\,,\,1-r)$ is the idempotent complement of e.

Proof. The triangle Δ is just a special type of diagram in $\mathcal{C}$. Since Σ is additive, it is clear that $\mathrm{im}(\Sigma p) = \Sigma(\mathrm{im}\,p)$ and the two direct summands of Δ produced by Remark 1.9 are the ones of the statement. $\square$

2. Monads, rings and modules

We review the notions of monad, ring objects and modules and refer the reader to Mac Lane [18]. Simultaneously, we adapt the terminology to the presence of a suspension (Def. 1.1), by requiring the structures to be "stable".

2.1. *Definition.* Let $\mathcal{C}$ be a category. A *monad* on $\mathcal{C}$ is a triple (M, μ, η), often just written M, where $M : \mathcal{C} \to \mathcal{C}$ is an endofunctor, $\mu : M^2 \to M$ (the *multiplication*) and $\eta : \mathrm{Id}_{\mathcal{C}} \to M$ (the *unit*) are natural transformations such that the following diagrams, expressing associativity and two-sided unit, commute:

(2.2)
$$\begin{tikzcd} M^3 \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "M\eta"] \arrow[dr, equal] & M^2 \arrow[d, "\mu"] & M \arrow[l, "\eta M"'] \arrow[dl, equal] \\ & M\,. & \end{tikzcd}$$

When the category $\mathcal{C}$ is suspended, we say that an additive monad M is *stable* if M, μ and η commute with suspension: $\Sigma M = M\Sigma$, $\mu\Sigma = \Sigma\mu$, $\eta\Sigma = \Sigma\eta$.

2.3. *Example.* Let $\mathcal{C}$ be a monoidal category with tensor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit $\mathbb{1} \in \mathcal{C}$, see [18, Chap. VII]. A (unital and associative) ring object in $\mathcal{C}$ is a triple (A, μ, η) where A is an object of $\mathcal{C}$ and where the *multiplication* $\mu : A \otimes A \to A$ and the *unit* $\eta : \mathbb{1} \to A$ are morphisms in $\mathcal{C}$ satisfying the usual associativity and two-sided unit conditions analogous to (2.2). Let $M : \mathcal{C} \to \mathcal{C}$ be the functor $A \otimes -$, with the obvious $\mu : M^2 \to M$ and $\eta : \mathrm{Id} \to M$. Then M is a monad on $\mathcal{C}$.

Of course, for R a commutative ring and for $\mathcal{C} = R\text{--}\mathrm{Mod}$ with $\otimes = \otimes_R$, the ring objects in $\mathcal{C}$ are the usual R-algebras. In particular, ring objects in $\mathbb{Z}\text{--}\mathrm{Mod}$ are ordinary rings. So, monads are generalizations of rings and algebras.

2.4. *Definition.* Let $M : \mathcal{C} \to \mathcal{C}$ be a monad. The *Eilenberg-Moore category of (left) M-modules* $M\text{--}\mathrm{Mod}_{\mathcal{C}}$ is defined as follows. A left M-module is a pair (x, λ) where

x is an object of $\mathcal{C}$ (the *underlying object*) and $\lambda : Mx \to x$ is a morphism (the *left action*) such that the following diagrams both commute:

$$\begin{tikzcd} M^2x \arrow[r, "M\lambda"] \arrow[d, "\mu_x"'] & Mx \arrow[d, "\lambda"] \\ Mx \arrow[r, "\lambda"'] & x \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, equal] & Mx \arrow[d, "\lambda"] \\ & x\,. \end{tikzcd}$$

A morphism $f : (x, \lambda) \to (x', \lambda')$ of left M-modules is a morphism $f : x \to x'$ in $\mathcal{C}$ which is M-linear, i.e. such that $\lambda' \circ M(f) = f \circ \lambda$. There is a *free module* functor $F_M : \mathcal{C} \to M\text{–Mod}_{\mathcal{C}}$ defined by $F_M(y) := \big(M(y), \mu_y\big)$. It has a right adjoint $G_M : M\text{–Mod}_{\mathcal{C}} \to \mathcal{C}$ which forgets the action. See [9]. We define *the Kleisli category of free M-modules $M\text{–Free}_{\mathcal{C}}$* as the full subcategory $F_M(\mathcal{C})$ of $M\text{–Mod}_{\mathcal{C}}$; see [12]. ([1]) The above functors F_M and G_M restrict to an adjunction between $\mathcal{C}$ and $M\text{–Free}_{\mathcal{C}}$:

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, shift right, "F_M"', "\text{(free module)}"'] \arrow[dr, shift right, "F_M"'] & \\ M\text{–Free}_{\mathcal{C}} \arrow[ur, shift right, "G_M"'] \arrow[rr, hook, "\text{(fully faithful)}"'] & & M\text{–Mod}_{\mathcal{C}}\,. \arrow[ul, shift right, "G_M"', "\text{(forget action)}"] \end{tikzcd}$$

When $\mathcal{C}$ is suspended and M is stable, then both $M\text{–Mod}_{\mathcal{C}}$ and $M\text{–Free}_{\mathcal{C}}$ inherit an obvious suspension such that F_M and G_M commute with suspension.

2.5. *Example.* When $A = (A, \mu, \eta)$ is a ring object in a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ as in Example 2.3, the above constructions yield the natural categories of left A-modules and free left A-modules (still relatively to the ambient $\mathcal{C}$).

2.6. *Remark.* For M additive, if $\mathcal{C}$ is idempotent-complete then so is $M\text{–Mod}_{\mathcal{C}}$.

2.7. *Remark.* Given an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$, let $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ be the unit and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ the counit of this adjunction. Then $M := GF$ is a monad on $\mathcal{C}$, with unit η and multiplication $\mu := G\epsilon F : M^2 = G(FG)F \xrightarrow{G\epsilon F} GF = M$. One says that M is *realized by the adjunction* (F, G). Given a monad $M : \mathcal{C} \to \mathcal{C}$, there are in general many adjunctions realizing M. They form a category in which the Kleisli construction $M\text{–Free}_{\mathcal{C}}$ is initial and the Eilenberg-Moore $M\text{–Mod}_{\mathcal{C}}$ is final:

2.8. **Proposition** ([18, Thm. VI.5.3])**.** *Let $M : \mathcal{C} \to \mathcal{C}$ be a monad realized by an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$. Then there are unique functors L and K as follows:*

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, shift right, "F_M"'] \arrow[d, shift right, "F"'] \arrow[dr, shift right, "F_M"'] & \\ M\text{–}Free_{\mathcal{C}} \arrow[ur, shift right, "G_M"'] \arrow[r, dashed, "\exists\,!\,L"'] & \mathcal{D} \arrow[u, shift right, "G"'] \arrow[r, dashed, "\exists\,!\,K"'] & M\text{–}Mod_{\mathcal{C}} \arrow[ul, shift right, "G_M"'] \end{tikzcd}$$

such that $F = L \circ F_M$, $G \circ L = G_M$, $G = G_M \circ K$ and $K \circ F = F_M$. Moreover, L is fully faithful. Finally, if $\mathcal{C}$ and $\mathcal{D}$ are suspended, if M is stable and if (F, G) is an adjunction of functors commuting with suspension, then the functors L and K commute with suspension as well.

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[1]In [18], $M\text{–Mod}_{\mathcal{C}}$ and $M\text{–Free}_{\mathcal{C}}$ are denoted $\mathcal{C}^M$ and $\mathcal{C}_M$ and called (free) M-algebras.

Proof. See the reference [18]. Compatibility with suspension is an easy exercise. ☐

2.9. *Remark.* If we assume moreover that $\mathcal{C}$ is (pre-) triangulated and that the stable monad $M:\mathcal{C}\to\mathcal{C}$ is an exact functor, it is legitimate to wonder whether M can be realized by an adjunction $F:\mathcal{C}\leftrightarrows\mathcal{D}:G$ in which $\mathcal{D}$ is also (pre-) triangulated and F and G are exact. This seems a difficult problem in general but if one requires G faithful, in the spirit of M-modules, then there is essential *at most one* solution, as we explain now. This is very probably in the literature already but it is worth observing it here anyway.

2.10. **Proposition.** *Let $\mathcal{C}$ be an additive category, $F:\mathcal{C}\leftrightarrows\mathcal{D}:G$ an adjunction with faithful right adjoint G and with $\mathcal{D}$ pre-triangulated. Let $M=GF:\mathcal{C}\to\mathcal{C}$ be the corresponding monad. Then the fully faithful functor $L:M\text{–}\mathit{Free}_{\mathcal{C}}\longrightarrow\mathcal{D}$ of Proposition 2.8 is moreover $\oplus$-cofinal (a. k. a. dense), that is, every object of $\mathcal{D}$ is a direct summand of the image by L of an object of $M\text{–}\mathit{Free}_{\mathcal{C}}$. In other words, L induces an equivalence between idempotent completions $L^\natural:(M\text{–}\mathit{Free}_{\mathcal{C}})^\natural\xrightarrow{\sim}\mathcal{D}^\natural$.*

Proof. For $x\in\mathcal{D}$, the counit $\epsilon_x:FGx\to x$ fits in a distinguished triangle $FGx\xrightarrow{\epsilon_x}x\xrightarrow{\varphi}y\longrightarrow\Sigma FGx$ in $\mathcal{D}$. So, $\varphi\epsilon_x=0$ hence $G(\varphi)G(\epsilon_x)=0$. But $G(\epsilon_x)$ is split surjective (one of the unit-counit relations) hence $G(\varphi)=0$. Since we assume G faithful, we get $\varphi=0$ which implies that ϵ_x is split surjective already in the pre-triangulated category $\mathcal{D}$. Hence x is a direct summand of $FGx=L(F_M Gx)$. ☐

2.11. *Definition.* We could call the idempotent completion $(M\text{–}\mathrm{Free}_{\mathcal{C}})^\natural$ the category of *projective M-modules (relatively to $\mathcal{C}$)* and denote it $M\text{–}\mathrm{Proj}_{\mathcal{C}}$.

3. Separability

Again, we start by recalling standard terminology, *cum grano salis.*

3.1. *Definition.* A ring object $A=(A,\mu,\eta)$ in a monoidal category $\mathcal{C}=(\mathcal{C},\otimes,\mathbb{1})$ is called *separable* if multiplication $\mu:A\otimes A\to A$ admits a section as two-sided A-module, i.e. a morphism $\sigma:A\to A\otimes A$ such that $\mu\sigma=\mathrm{id}_A$ and such that the following diagram commutes:

$$
\begin{tikzcd}
& A\otimes A \arrow[dl, "\sigma\otimes 1"'] \arrow[d, "\mu"] \arrow[dr, "1\otimes\sigma"] & \\
A\otimes A\otimes A \arrow[dr, "1\otimes\mu"'] & A \arrow[d, "\sigma"] & A\otimes A\otimes A \arrow[dl, "\mu\otimes 1"] \\
& A\otimes A\,. &
\end{tikzcd}
\tag{3.2}
$$

3.3. *Remark.* This is the usual definition of separable R-algebra when $\mathcal{C}=R\text{−}\mathrm{Mod}$ and R is commutative; see DeMeyer-Ingraham [8, § II.1] or Knus-Ojanguren [13, § III.1]. (Over a field K, a commutative K-algebra A is sometimes called "(classically) separable" if $L\otimes_K A$ is reduced for every extension L/K. An algebra over K is separable if and only if it is "classically separable" and has finite dimension as a vector space over K. See [8, Thm. 2.5].) In general, our separable R-algebras need not be finitely generated as R-modules, nor commutative.

Over the integers, there exists no commutative separable $\mathbb{Z}$-algebra A, which is finitely generated projective as a $\mathbb{Z}$-module, except of course the trivial ones $\mathbb{Z} \times \cdots \times \mathbb{Z}$; see Example 6.2 below. The analogous result in topological stable homotopy theory is due to Rognes, see [23, Thm. 1.3, p. 4], who proved that there is no commutative Galois ring object in $\mathrm{SH}^{\mathrm{fin}}$, the stable homotopy category of finite CW-complexes, beyond the obvious $\mathbb{1} \oplus \cdots \oplus \mathbb{1}$; see Example 6.2 below.

These are not negative results. First of all, we do not assume our rings to be finitely generated, nor commutative. But even under these assumptions, the theory of separability should be understood as a *relative notion* as illustrated both in algebraic geometry by the well-known importance of étale algebras and in topology by the many examples to be found in the reference [23] above.

3.4. *Example.* Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a tensor triangulated category and let $x \in \mathcal{C}$ be a *rigid* object, i.e. with an adjunction $x \otimes - : \mathcal{C} \leftrightarrows \mathcal{C} : Dx \otimes -$ for some $Dx \in \mathcal{C}$. Let us denote by $\eta : \mathbb{1} \to Dx \otimes x$ and $\epsilon : x \otimes Dx \to \mathbb{1}$ the associated unit and counit. Consider the ring object $A = \underline{\mathrm{end}}(x) := Dx \otimes x$ in $\mathcal{C}$ with multiplication

$$\mu : \quad A \otimes A = Dx \otimes x \otimes Dx \otimes x \xrightarrow{\;1 \otimes \epsilon \otimes 1\;} Dx \otimes \mathbb{1} \otimes x \cong A\,.$$

Suppose now that x is *faithful*, that is, $x \otimes - : \mathcal{C} \to \mathcal{C}$ is faithful. This is equivalent to say that $\epsilon : x \otimes Dx \to \mathbb{1}$ is split surjective (as in the proof of Proposition 2.10). Choose a section $\sigma_0 : \mathbb{1} \to x \otimes Dx$ of ϵ. Then the morphism

$$\sigma : \quad A \cong Dx \otimes \mathbb{1} \otimes x \xrightarrow{\;1 \otimes \sigma_0 \otimes 1\;} Dx \otimes x \otimes Dx \otimes x = A \otimes A$$

is a section of μ, which satisfies (3.2). In short, for x rigid and faithful, $\underline{\mathrm{end}}(x)$ is separable. (Note that, in most conventions, the above multiplication on $\underline{\mathrm{end}}(x)$ is rather the *opposite* of the one induced by "composition". However, a ring object is separable if and only if its opposite is separable.)

Generalizing the notion of separability to monads gives (see [7, 6.3] or [6, 2.9]) :

3.5. *Definition.* A monad $M : \mathcal{C} \to \mathcal{C}$ is called *separable* if $\mu : M^2 \to M$ admits a section $\sigma : M \to M^2$, i.e. $\mu \circ \sigma = \mathrm{id}_M$, satisfying the analogue of (3.2), that is :

$$M\mu \circ \sigma M = \sigma \circ \mu = \mu M \circ M\sigma\,. \tag{3.6}$$

If moreover $\mathcal{C}$ is suspended (Def. 1.1) and M is stable (Def. 2.1) then we say that M is *stably separable* if σ commutes with suspension as well.

On the other hand, there is an a priori unrelated notion of *separable functor*, which could be understood as being "split faithful" (see Năstăsescu et. al. [20, § 1]) :

3.7. *Definition.* A functor $G : \mathcal{D} \to \mathcal{C}$ is *separable* if there exist retractions

$$H = H_{x,y} : \mathcal{C}(Gx, Gy) \longrightarrow \mathcal{D}(x, y) \tag{3.8}$$

of the maps induced by G on morphisms, which are natural in $x, y \in \mathcal{D}$. This means that $H(G(f)) = f$ for every morphism f in $\mathcal{D}$ and that $H\big(G(g)\,k\,G(f)\big) = g\,H(k)\,f$ for all morphisms k in $\mathcal{C}$ and f, g in $\mathcal{D}$ for which the composition makes sense.

When $\mathcal{C}$ and $\mathcal{D}$ are suspended (Def. 1.1) and G commutes with the suspension, we say that G is *stably separable* if H moreover commutes with suspension, meaning that $H_{\Sigma x, \Sigma y} \circ \Sigma = \Sigma \circ H_{x,y}$.

3.9. *Remark.* If $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, Rafael [22, Thm. 1.2] proved that G is separable if and only if the counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ has a section

$$\xi : \mathrm{Id}_{\mathcal{D}} \longrightarrow F\,G \tag{3.10}$$

i.e. a natural transformation ξ such that $\epsilon \circ \xi = \mathrm{id}$. The dictionary between H and ξ is given by the following formulas for every $x, y \in \mathcal{D}$ and $k : Gx \to Gy$ in $\mathcal{C}$:

$$\xi_x = H_{x,FGx}(\eta_{Gx} : Gx \to GFGx) \qquad \text{and} \qquad H_{x,y}(k) = \epsilon_y \circ Fk \circ \xi_x \,.$$

In the suspended situation, it is then easy to see that G is *stably* separable if and only if there exists such a ξ which commutes with suspension.

One does *not* define a separable monad M by requiring M to be separable as a functor! These two notions of separability are actually related as follows:

3.11. **Proposition** ([6, 2.9 (1)] or [7, Prop. 6.3])**.** *Let $M : \mathcal{C} \to \mathcal{C}$ be a monad. Then M is a separable monad if and only if the forgetful functor $G_M : M\text{–}Mod_{\mathcal{C}} \longrightarrow \mathcal{C}$ is a separable functor. Moreover, if we assume that $\mathcal{C}$ is a suspended category, then M is a* stably *separable monad if and only G_M is a* stably *separable functor.*

Proof. Since the forgetful functor G_M has a left adjoint $F_M : \mathcal{C} \longrightarrow M\text{–}\mathrm{Mod}_{\mathcal{C}}$, its (stable) separability is equivalent to the existence of a (stable) section $\xi : \mathrm{Id}_{M\text{–}\mathrm{Mod}_{\mathcal{C}}} \longrightarrow F_M G_M$ of the unit $\epsilon : F_M G_M \to \mathrm{Id}_{M\text{–}\mathrm{Mod}_{\mathcal{C}}}$ as in Remark 3.9. Indeed, there is an explicit dictionary between such sections ξ and the sections $\sigma : M \to M^2$ of the monad's multiplication, as in Definition 3.5. To ξ corresponds $\sigma : M = G_M F_M \xrightarrow{G_M \xi F_M} G_M F_M G_M F_M = M^2$. Conversely, to σ corresponds

$$\xi_{(x,\lambda)} : \quad x \xrightarrow{\eta_x} M(x) \xrightarrow{\sigma_x} M^2(x) \xrightarrow{M(\lambda)} M(x) = F_M G_M(x, \lambda)$$

for every M-module (x, λ). The equivalence is proved in the references. This correspondence preserves "stability", hence the second part of the statement. $\square$

4. Pre-triangulation on the category of modules

4.1. **Theorem.** *Let $\mathcal{C}$ be a pre-triangulated category and let $\mathcal{D}$ is an idempotent-complete suspended category. Let*

$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "F"'] \\ \mathcal{D} \arrow[u, shift right, "G"'] \end{tikzcd}$$

be an adjunction (F left adjoint and G right adjoint) of functors commuting with suspension. Suppose that the stable monad $GF : \mathcal{C} \to \mathcal{C}$ is exact and that $G : \mathcal{D} \to \mathcal{C}$ is a stably separable functor (Def. 3.7). Then $\mathcal{D}$ is pre-triangulated with distinguished triangles Δ being exactly the ones such that $G(\Delta)$ is distinguished in $\mathcal{C}$. Moreover, with this pre-triangulation both functors F and G become exact.

Proof. The reader is referred to Definition 3.7 and Remark 3.9 for separability of G. In particular, we use the notation H for the retraction of G on morphisms as in (3.8) and $\xi : \mathrm{Id}_{\mathcal{D}} \to FG$ for the section of the counit ϵ, as in (3.10).

Let us verify (TC 2.1)-(TC 2.3) of Definition 1.3. The bookkeeping axioms (TC 2.1) are easily verified by applying G to the triangles which are candidate for distinction and by using the corresponding axioms in $\mathcal{C}$. The main difficulty will be the existence of distinguished triangles over every morphism (TC 2.2). Indeed, the morphism axiom (TC 2.3) is easy, as we now verify. Consider two distinguished triangles in $\mathcal{D}$ and the beginning of a morphism (α, β) :

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\alpha"'] & y \arrow[r, "g"] \arrow[d, "\beta"'] & z \arrow[r, "h"] \arrow[d, dotted, "\gamma"'] & \Sigma x \arrow[d, "\Sigma\alpha"] \\ x' \arrow[r, "f'"'] & y' \arrow[r, "g'"'] & z' \arrow[r, "h'"'] & \Sigma x' . \end{tikzcd}$$

Applying G to this diagram, we get a similar diagram to which we can apply the morphism axiom in $\mathcal{C}$ to produce some fill-in map $\tilde{\gamma} : G(z) \to G(z')$. Then setting $\gamma := H_{z,z'}(\tilde{\gamma})$ yields a fill-in map as wanted. For instance, $\gamma\, g = H(\tilde{\gamma})g = H(\tilde{\gamma}\, G(g)) = H(G(g')\, G(\beta)) = H(G(g'\beta)) = g'\beta$ and similarly for the other square.

We now have (TC 2.1) and (TC 2.3) for $\mathcal{D}$. Let us finally prove axiom (TC 2.2). Let $f : x \to y$ be a morphism in $\mathcal{D}$ and consider a distinguished triangle $\hat{\Delta}$ with base $G(f)$ in the pre-triangulated category $\mathcal{C}$:

$$\hat{\Delta} = \Big(\; Gx \xrightarrow{G(f)} Gy \xrightarrow{\hat{g}} \hat{z} \xrightarrow{\hat{h}} \Sigma Gx \; \Big) .$$

Using naturality of $\xi : \mathrm{Id}_{\mathcal{D}} \to FG$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, as well as $\epsilon\,\xi = \mathrm{Id}_{\mathcal{D}}$, we see that the morphism $f : x \to y$ is a direct summand of the morphism $FG(f)$ in $\mathcal{D}$:

$$F(\hat{\Delta}) : \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\xi_x"'] & y \arrow[r, dotted] \arrow[d, "\xi_y"'] & ? \arrow[r, dotted] \arrow[d, dotted] & \Sigma x \arrow[d, "\Sigma\xi_x"] \\ FGx \arrow[r, "FG(f)"] \arrow[d, "\epsilon_x"'] & FGy \arrow[r, "F(\hat{g})"] \arrow[d, "\epsilon_y"'] & F\hat{z} \arrow[r, "F(\hat{h})"] \arrow[d, dotted] & \Sigma FGx \arrow[d, "\Sigma\epsilon_x"] \\ x \arrow[r, "f"] & y \arrow[r, dotted] & ? \arrow[r, dotted] & \Sigma x . \end{tikzcd}$$

The triangle $F(\hat{\Delta})$ is distinguished in $\mathcal{D}$ since GF is exact. We now want to construct a direct summand of $F\hat{z}$ and a triangle with base f which will be a direct summand of $F(\hat{\Delta})$. By Lemma 1.6 (d) applied to $\mathcal{D}$, there is an idempotent $e = (\xi_x\epsilon_x\,,\, \xi_y\epsilon_y\,,\, r) = e^2$ in $\mathcal{D}$ of the distinguished triangle $F(\hat{\Delta})$:

$$\begin{tikzcd} FGx \arrow[r, "FG(f)"] \arrow[d, "\xi_x\epsilon_x"'] & FGy \arrow[r, "F(\hat{g})"] \arrow[d, "\xi_y\epsilon_y"'] & F\hat{z} \arrow[r, "F(\hat{h})"] \arrow[d, dotted, "\exists"', "r=r^2"] & \Sigma FGx \arrow[d, "\Sigma(\xi_x\epsilon_x)"] \\ FGx \arrow[r, "FG(f)"'] & FGy \arrow[r, "F(\hat{g})"'] & F\hat{z} \arrow[r, "F(\hat{h})"'] & \Sigma FGx . \end{tikzcd}$$

Since $\mathcal{D}$ is idempotent-complete, Proposition 1.10 gives a decomposition $F(\hat{\Delta}) = \Delta \oplus \Delta'$, for triangles Δ and Δ' corresponding to the idempotents e and $1 - e$ respectively. By construction, the summand Δ corresponding to e has the form

$$\Delta = \Big(\; x \xrightarrow{f} y \xrightarrow{g} \mathrm{im}(r) \xrightarrow{h} \Sigma x \; \Big)$$

where $g = rF(\hat{g})\xi_y$ and $h = \Sigma(\epsilon_x)F(\hat{h})r$. Then $G(\Delta)$ is a direct summand of the triangle $GF(\hat{\Delta})$ which is distinguished in $\mathcal{C}$ since GF is exact. A direct summand of a distinguished triangle in the pre-triangulated category $\mathcal{C}$ is distinguished, see [21, Prop. 1.2.3]. So, $G(\Delta)$ is distinguished in $\mathcal{C}$ and Δ is distinguished in $\mathcal{D}$. $\square$

4.2. *Remark.* Note that we did not require the functor G to be full, in which case F would be a Bousfield localization by [14, Prop. 4.9.1] for instance. See more in Example 6.3 below.

4.3. **Corollary.** *Let $\mathcal{C}$ be an idempotent-complete pre-triangulated category and let $M : \mathcal{C} \to \mathcal{C}$ be an exact monad. Suppose that M is a stably separable monad (Def. 3.5). Then every M-module is projective (relatively to $\mathcal{C}$): $M\text{–Proj}_{\mathcal{C}} = M\text{–Mod}_{\mathcal{C}}$, i.e. the Eilenberg-Moore category $M\text{–Mod}_{\mathcal{C}}$ is the idempotent completion of the Kleisli category $M\text{–Free}_{\mathcal{C}}$. More important, $M\text{–Mod}_{\mathcal{C}}$ admits a pre-triangulation such that*

(a) the free-module functor $F_M : \mathcal{C} \to M\text{–Mod}_{\mathcal{C}}$ is exact,

(b) the forgetful functor $M\text{–Mod}_{\mathcal{C}} \to \mathcal{C}$ is exact.

In fact, the pre-triangulation is characterized by any of the properties (a) or (b).

Proof. Let $\mathcal{D} = M\text{–Mod}_{\mathcal{C}}$. We claim that the free-module/forgetful adjunction $F_M : \mathcal{C} \leftrightarrows \mathcal{D} : G_M$ of Definition 2.4 satisfies the assumptions of Theorem 4.1. Indeed, $\mathcal{D} = M\text{–Mod}_{\mathcal{C}}$ is an idempotent-complete suspended category. By Proposition 3.11, G_M is a stably separable functor. Finally, $G_M F_M = M$ is exact. Then Theorem 4.1 yields a pre-triangulation on $\mathcal{D}$ and Proposition 2.10 gives $(M\text{–Free}_{\mathcal{C}})^\natural = \mathcal{D}^\natural = \mathcal{D}$. Uniqueness of the triangulation is easily left to the reader. $\square$

5. Octahedron and higher triangulations

The study of n-triangles for $n \geq 1$, or higher octahedra, was initiated in Beilinson et. al. [5, Rem. 1.1.14]. A 1-triangle is just the data of an object, a 2-triangle is a good old triangle and a 3-triangle is an octahedron. Let us review this with the goal of introducing Künzer's higher axiomatic [15]. See also Maltsiniotis [19]. Since these references are still somewhat confidential at this stage, we provide explanations, pictures and examples, to help the reader get acquainted with these objects.

5.1. *Definition.* Let $(\mathcal{C}, \Sigma)$ be a suspended category (Def. 1.1). Let $n \geq 1$. An *n-triangle* Θ is defined as a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} a_{i,j} \arrow[r, "f_{i,j}"] \arrow[d, "g_{i,j}"'] & a_{i,j+1} \arrow[d, "g_{i,j+1}"] \\ a_{i+1,j} \arrow[r, "f_{i+1,j}"'] & a_{i+1,j+1} \end{tikzcd}$$

with objects $a_{i,j}$ indexed by $(i,j) \in \mathbb{Z}^2$ and morphisms $f_{i,j}$ and $g_{i,j}$ as above, subject to the following rules:

(i) the diagram lives in a diagonal strip: $a_{i,j} = 0$ unless $1 \leq j - i \leq n$,

(ii) the diagram has periodicity: for all $(i,j) \in \mathbb{Z}^2$, we have $a_{i,j+n+1} = \Sigma(a_{j,i})$ on objects, whereas $f_{i,j+n+1} = \Sigma(g_{j,i})$ and $g_{i,j+n+1} = \Sigma(f_{j,i})$ on morphisms.

So, all information contained in Θ is exactly in the finite commutative diagram

$$
\begin{tikzcd}
a_{0,1} \arrow[r, "f_{0,1}"] \arrow[d] & a_{0,2} \arrow[r, "f_{0,2}"] \arrow[d, "g"] & a_{0,3} \arrow[r, "f_{0,3}"] \arrow[d, "g"] & \cdots \arrow[r, "f_{0,n-2}"] & a_{0,n-1} \arrow[r, "f_{0,n-1}"] \arrow[d, "g"'] & a_{0,n} \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\
0 \arrow[r] & a_{1,2} \arrow[r, "f"] \arrow[d] & a_{1,3} \arrow[r, "f"] \arrow[d, "g"] & \cdots \arrow[r, "f"] & a_{1,n-1} \arrow[r, "f"] \arrow[d, "g"'] & a_{1,n} \arrow[r, "f"] \arrow[d, "g"] & \Sigma a_{0,1} \arrow[d, "\Sigma f_{0,1}"] \\
 & 0 \arrow[r] & a_{2,3} \arrow[r, "f"] & \cdots \arrow[r, "f"] & a_{2,n-1} \arrow[r, "f"] \arrow[d, "g"'] & a_{2,n} \arrow[r, "f"] \arrow[d, "g"'] & \Sigma a_{0,2} \arrow[d, "\Sigma f_{0,2}"] \\
 & & \ddots & \ddots & \vdots \arrow[d, "g"'] & \vdots \arrow[d, "g"'] & \vdots \arrow[d, "\Sigma f_{0,n-3}"] \\
 & & & 0 \arrow[r] & a_{n-2,n-1} \arrow[r, "f"] \arrow[d] & a_{n-2,n} \arrow[r, "f"] \arrow[d, "g"'] & \Sigma a_{0,n-2} \arrow[d, "\Sigma f_{0,n-2}"] \\
 & & 0 & & 0 \arrow[r] & a_{n-1,n} \arrow[r, "f"] \arrow[d] & \Sigma a_{0,n-1} \arrow[d, "\Sigma f_{0,n-1}"] \\
 & & & & & 0 \arrow[r] & \Sigma a_{0,n}\,.
\end{tikzcd}
\tag{5.2}
$$

Rule (ii) then means that the right-hand column is the suspension of the top row

$$
a_{0,1} \xrightarrow{f_{0,1}} a_{0,2} \xrightarrow{f_{0,2}} a_{0,3} \xrightarrow{f_{0,3}} \cdots \longrightarrow a_{0,n-1} \xrightarrow{f_{0,n-1}} a_{0,n}\,.
$$

That top row is called the *base* of the n-triangle Θ.

5.3. *Remark.* These n-triangles Θ should remind the reader of the n-simplices in Waldhausen's $S.$-construction [25], in which the base is composed of admissible monomorphisms and $a_{i,j}$ is the quotient $a_{0,j}/a_{0,i}$. Here, we pretend instead that for every $1 \le i < j \le n$ the following triangle is distinguished:

$$
\begin{tikzcd}
a_{0,i} \arrow[r] & a_{0,j} \arrow[d] & \\
 & a_{i,j} \arrow[r] & \Sigma(a_{0,i})\,.
\end{tikzcd}
\tag{5.4}
$$

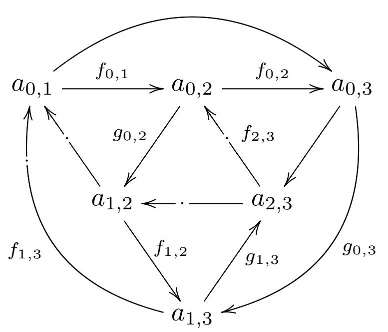
This replaces Waldhausen's choice of an exact sequence $a_{0,i} \rightarrowtail a_{0,j} \twoheadrightarrow a_{i,j}$.

Waldhausen's construction is simplicial and the same holds here. The face operation d_k is very easy: It removes all objects $a_{i,j}$ with i or j congruent to k modulo $n+1$ (and composes morphisms over the gap). The degeneracy operation s_k is easy too: For $0 \le k \le n$, it repeats the object $a_{0,k}$ in the extended base: $0 = a_{0,0} \to a_{0,1} \to \cdots \to a_{0,k} \xrightarrow{1} a_{0,k} \to \cdots \to a_{0,n}$. The effect of s_k on the rest of the n-triangle is controlled by the rule (5.4), followed in the most natural way. So, one has to include zero objects (=cones of identity morphisms) and identity morphisms at the relevant places. A posteriori, one can forget about the above recipe (after all, there are no distinguished triangles yet) and describe the simplicial structure by formulas. This is done in our references [15, 16, 19]. We explicitly unfold the case $n = 3$ below.

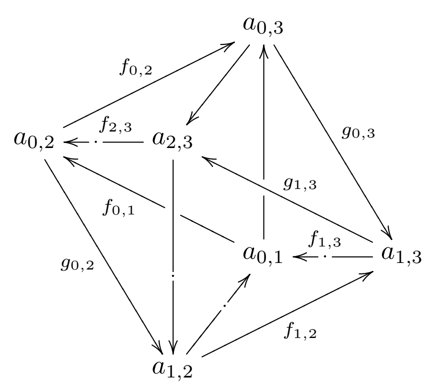
5.5. *Example.* A 2-triangle is just a triangle, as in (1.2) – on the right. A 3-triangle

$$
\Omega = \left(\begin{tikzcd}
a_{0,1} \arrow[r, "f_{0,1}"] & a_{0,2} \arrow[r, "f_{0,2}"] \arrow[d, "g_{0,2}"] & a_{0,3} \arrow[d, "g_{0,3}"] & \\
& a_{1,2} \arrow[r, "f_{1,2}"'] & a_{1,3} \arrow[r, "f_{1,3}"] \arrow[d, "g_{1,3}"] & \Sigma a_{0,1} \arrow[d, "\Sigma f_{0,1}"] \\
& & a_{2,3} \arrow[r, "f_{2,3}"'] & \Sigma a_{0,2} \arrow[d, "\Sigma f_{0,2}"] \\
& & & \Sigma a_{0,3}
\end{tikzcd}\right) \tag{5.6}
$$

is usually called an *octahedron*, often presented as



or



The four faces which are not triangles are commutative, hence defining the four arrows which are not named in (5.6), like $a_{2,3} \longrightarrow \Sigma a_{1,2}$ which must be $\Sigma(g_{0,2})\, f_{2,3}$. In the simplicial structure à *la* Waldhausen, every triangle $\Delta = \left(a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a \right)$ yields three *degeneracies*, namely the following octahedra $s_0(\Delta)$, $s_1(\Delta)$ and $s_2(\Delta)$:

$$
\begin{tikzcd}
0 \arrow[r] & a \arrow[r, "f"] \arrow[d, equal] & b \arrow[d, equal] & \\
& a \arrow[r, "f"'] & b \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\
& & c \arrow[r, "h"'] & \Sigma a \arrow[d, "\Sigma f"] \\
& & & \Sigma b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, equal, "1"] & a \arrow[r, "f"] \arrow[d] & b \arrow[d, "g"] & \\
& 0 \arrow[r] & c \arrow[r, "h"'] \arrow[d, equal] & \Sigma a \arrow[d, equal] \\
& & c \arrow[r, "h"'] & \Sigma a \arrow[d, "\Sigma f"] \\
& & & \Sigma b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[r, equal, "1"] \arrow[d, "g"'] & b \arrow[d, "g"] & \\
& c \arrow[r, equal] & c \arrow[r, "h"'] \arrow[d] & \Sigma a \arrow[d, "\Sigma f"] \\
& & 0 \arrow[r] & \Sigma b \arrow[d, equal] \\
& & & \Sigma b\,.
\end{tikzcd}
$$

And to every octahedron Ω as in (5.6), we can associate four *faces*, i.e. the triangles:

$$
\begin{aligned}
d_3(\Omega) &= \Big(\; a_{0,1} \xrightarrow{\;f_{0,1}\;} a_{0,2} \xrightarrow{\;g_{0,2}\;} a_{1,2} \xrightarrow{\;f_{1,3}\,f_{1,2}\;} \Sigma a_{0,1} \;\Big)\\
d_2(\Omega) &= \Big(\; a_{0,1} \xrightarrow{\;f_{0,2}\,f_{0,1}\;} a_{0,3} \xrightarrow{\;g_{0,3}\;} a_{1,3} \xrightarrow{\;f_{1,3}\;} \Sigma a_{0,1} \;\Big)\\
d_1(\Omega) &= \Big(\; a_{0,2} \xrightarrow{\;f_{0,2}\;} a_{0,3} \xrightarrow{\;g_{1,3}\,g_{0,3}\;} a_{2,3} \xrightarrow{\;f_{2,3}\;} \Sigma a_{0,2} \;\Big)\\
d_0(\Omega) &= \Big(\; a_{1,2} \xrightarrow{\;f_{1,2}\;} a_{1,3} \xrightarrow{\;g_{1,3}\;} a_{2,3} \xrightarrow{\;\Sigma(g_{0,2})\,f_{2,3}\;} \Sigma a_{1,2} \;\Big).
\end{aligned}
\tag{5.7}
$$

5.8. *Remark.* A pre-triangulated category $\mathcal{C}$ is called *triangulated in the sense of Verdier* if any pair of composable morphisms $a_{0,1} \to a_{0,2} \to a_{0,3}$ is the base of an octahedron Ω as in (5.6), whose four faces $\{d_i(\Omega)\}_{0\leq i\leq 3}$ as in (5.7) are distinguished triangles. The point is that Verdier *defines* his good octahedra by the distinction of their four faces. So, being triangulated is a property of a pre-triangulated category, not an additional structure. As announced, to adapt Theorem 4.1 and Corollary 4.3 beyond pre-triangulated categories, we need the stronger higher axiomatic of [15, 19], whose bookkeeping involves the following:

5.9. *Definition.* Given an n-triangle Θ, its *symmetric* $\sigma(\Theta)$ is the n-triangle obtained by applying Σ to every entry of Θ and changing the sign of every horizontal morphism in the last column:

$$
\sigma(\Theta) = \quad
\begin{tikzcd}
\Sigma a_{0,1} \arrow[r, "\Sigma f_{0,1}"] & \Sigma a_{0,2} \arrow[r, "\Sigma f_{0,2}"] \arrow[d, "\Sigma g"] & \cdots \arrow[r] & \Sigma a_{0,n-1} \arrow[r, "\Sigma f_{0,n-1}"] \arrow[d, "\Sigma g"'] & \Sigma a_{0,n} \arrow[d, "\Sigma g"] & \\
& \Sigma a_{1,2} \arrow[r, "\Sigma f"] & \cdots \arrow[r] & \Sigma a_{1,n-1} \arrow[r, "\Sigma f"] \arrow[d, "\Sigma g"'] & \Sigma a_{1,n} \arrow[r, "-\Sigma f"] \arrow[d, "\Sigma g"'] & \Sigma^2 a_{0,1} \arrow[d, "\Sigma^2 f_{0,1}"] \\
& & \ddots & \vdots \arrow[d, "\Sigma g"'] & \vdots \arrow[d, "\Sigma g"'] & \vdots \arrow[d, "\Sigma^2 f_{0,n-3}"] \\
& & & \Sigma a_{n-2,n-1} \arrow[r, "\Sigma f"] & \Sigma a_{n-2,n} \arrow[r, "-\Sigma f"] \arrow[d, "\Sigma g"'] & \Sigma^2 a_{0,n-2} \arrow[d, "\Sigma^2 f_{0,n-2}"] \\
& & & & \Sigma a_{n-1,n} \arrow[r, "-\Sigma f"] & \Sigma^2 a_{0,n-1} \arrow[d, "\Sigma^2 f_{0,n-1}"] \\
& & & & & \Sigma^2 a_{0,n}
\end{tikzcd}
$$

The name comes from the fact that the right-hand *column* of Θ in (5.2) now becomes the base *row* of $\sigma(\Theta)$. On the other hand, the *translate* $\tau(\Theta)$ is the n-triangle which has $a_{i+1,j+1}$ in place (i,j) and similarly for morphisms, without any sign.

5.10. *Example.* The symmetric and the translate of a 2-triangle $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ are the two triangles which appear in (TC 2.1.c) in Definition 1.3.

We can now formulate the higher triangulation in rather compact form.

5.11. *Definition.* Let $\mathcal{C}$ be a suspended category (Def. 1.1) and $N \geq 2$. We call *triangulation of order N on* $\mathcal{C}$ a collection of *distinguished n-triangles* (5.2) for all $n \leq N$ ([2]) such that the following axioms hold true for every $2 \leq n \leq N$:

(TC n.1) *Bookkeeping Axioms* :

- (TC n.1.a) Any n-triangle isomorphic to a distinguished one is distinguished.
- (TC n.1.b) Distinguished triangles are preserved by the simplicial structure *à la* Waldhausen, as explained in Remark 5.3 : Degeneracies of distinguished $(n-1)$-triangles are distinguished n-triangles and faces of distinguished n-triangles are distinguished $(n-1)$-triangles.
- (TC n.1.c) An n-triangle Θ is distinguished if and only if its symmetric $\sigma(\Theta)$ is distinguished; and Θ is distinguished if and only if its translate $\tau(\Theta)$ is distinguished. See Definition 5.9 above.

(TC n.2) *Existence Axiom* : Every $(n-1)$-tuple of composable morphisms is the base of a distinguished n-triangle.

(TC n.3) *Morphism Axiom* : Given two distinguished n-triangles, every morphism between their bases extends to a morphism of n-triangles.

A functor between triangulated categories of order $N \geq 2$ is *exact up to order N* if it commutes with suspension and preserves distinguished N-triangles (and a fortiori distinguished n-triangles for all $n \leq N$).

A category with triangulation of infinite order, i.e. distinguished n-triangles for all $n \in \mathbb{N}$ satisfying (TC n.1-3), is called *∞-triangulated.*

5.12. *Remark.* The homotopy category of a stable model category is ∞-triangulated. Actually, the value $\mathbb{D}(I)$ of a triangulated derivator $\mathbb{D}$ at any admissible category I is ∞-triangulated, see [19, Thm. 2]. So, morally speaking, all triangulated categories which appear in real life are ∞-triangulated.

5.13. *Remark.* It is clear that

$$\boxed{\begin{array}{c}\textit{Triangulation}\\ \textit{of third order}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\textit{Triangulation}\\ \textit{\`a la Verdier}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\textit{Triangulation}\\ \textit{of second order}\end{array}} = \boxed{\begin{array}{c}\textit{Pre-}\\ \textit{triangulation.}\end{array}}$$

One can actually give variants of these definitions, following Verdier, by looking at n-triangles $n \geq N+1$ whose faces are distinguished (up to some size). See [19].

5.14. *Remark.* As in Definition 1.3, axioms (TC n.2) and (TC n.3) win the juicy contest. Bookkeeping axioms are far from trivial, though, and one can actually add more of them, as in the strong form of the octahedron of [5, Rem. 1.1.13] or the *folding* of [16, 1.2.2.2]. Our references do not prove that the homotopy category of a stable model category satisfies these additional axioms, although it is expected. In any case, all such axioms easily pass from $\mathcal{C}$ to categories of M-modules.

5.15. *Remark.* When Θ is a distinguished n-triangle, an iterated application of the faces implies that every triangle as in (5.4) is distinguished. In other words, every $a_{i,j}$ for $1 \leq i < j \leq n$ is the cone of the morphism $a_{0,i} \to a_{0,j}$ composed from the base of Θ. We shall use this in the proof of the Main Theorem below.

[2]By convention, all 1-triangles are distinguished – these are just objects of $\mathcal{C}$.

5.16. *Remark.* We leave it to the reader to adapt the basic results of pre-triangulated categories to triangulated categories of order N, as in [21, §§ 1.1-1.2] with n-triangles, $n \le N$, instead of 2-triangles. In particular, for n-triangles Θ_1 and Θ_2, their sum $\Theta_1 \oplus \Theta_2$ is distinguished if and only if both Θ_1 and Θ_2 are distinguished.

5.17. **Main Theorem.** *Let $\mathcal{C}$ be an idempotent-complete category with a triangulation of order $N \ge 2$ (Def. 5.11) and let M be a stably separable monad on $\mathcal{C}$ (Def. 3.5) such that $M : \mathcal{C} \to \mathcal{C}$ is exact up to order N. Then:*

(a) All M-modules in $\mathcal{C}$ are projective (relatively to $\mathcal{C}$), i.e. the idempotent completion of the Kleisli category of free M-modules coincides with the Eilenberg-Moore category of M-modules: $(M\text{–}Free_{\mathcal{C}})^\natural = M\text{–}Mod_{\mathcal{C}}$.

(b) The category of M-modules $M\text{–}Mod_{\mathcal{C}}$ admits a triangulation of order N such that, for all $n \le N$, an n-triangle of M-modules Θ is distinguished exactly when the underlying n-triangle $G_M(\Theta)$ is distinguished in $\mathcal{C}$.

(c) Both the free-module functor $F_M : \mathcal{C} \to M\text{–}Mod_{\mathcal{C}}$ and the forgetful functor $G_M : M\text{–}Mod_{\mathcal{C}} \to \mathcal{C}$ are exact up to order N. Each of these properties characterizes the triangulation on $M\text{–}Mod_{\mathcal{C}}$.

(d) Let $\mathcal{D}$ be an idempotent-complete suspended category and $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ an adjunction of functors commuting with suspension (F left adjoint) which realizes M. Suppose that G is stably separable. Then the functors L and K

$$M\text{–}Free_{\mathcal{C}} \xrightarrow{L} \mathcal{D} \xrightarrow{K} M\text{–}Mod_{\mathcal{C}}$$

of Proposition 2.8 have the following properties: K is an equivalence and L is an equivalence after idempotent completion.

Proof. The category $\mathcal{C}$ is in particular pre-triangulated since $N \ge 2$. So, we can apply Corollary 4.3 to get (a) and Theorem 4.1 to show that $\mathcal{D}$ is pre-triangulated in (d). By Proposition 2.10 the functor $L^\natural : M\text{–}\mathrm{Free}_{\mathcal{C}}{}^\natural \longrightarrow \mathcal{D}^\natural = \mathcal{D}$ is an equivalence. By (a), $K \circ L^\natural$ is an equivalence, so the rest of (d) follows. Part (c) will follow from the proof of (b), which we treat now. The argument is essentially the same as in the proof of Theorem 4.1 and we only indicate the relevant modifications.

By Proposition 3.11, the forgetful functor $G_M : M\text{–}\mathrm{Mod}_{\mathcal{C}} \longrightarrow \mathcal{C}$ is stably separable (Def. 3.7). Let H be a retraction of G_M on morphisms as in (3.8) and $\xi : \mathrm{Id} \to F_M G_M$ be the section of the counit $\epsilon : F_M G_M \to \mathrm{Id}$, as in (3.10).

Let us verify (TC n.1)-(TC n.3) of Definition 5.11. The bookkeeping axioms are easy to verify by just applying G_M to the n-triangles which are candidate for distinction and by using the corresponding bookkeeping in $\mathcal{C}$. The morphism axiom (TC n.3) goes as (TC 2.3): We complete the given morphism between the bases in $\mathcal{C}$, after applying G_M, and we apply the retraction H to everything in sight to recover a morphism between the original n-triangles in $M\text{–}\mathrm{Mod}_{\mathcal{C}}$. Finally, for (TC n.2), let $a_{0,1} \xrightarrow{f_{0,1}} \cdots \xrightarrow{f_{0,n-1}} a_{0,n}$ in $M\text{–}\mathrm{Mod}_{\mathcal{C}}$ be an $(n-1)$-tuple of composable morphisms. Complete its image $G_M(a_{0,1}) \xrightarrow{G_M(f_{0,1})} \cdots \xrightarrow{G_M(f_{0,n-1})} G_M(a_{0,n})$ into a distinguished n-triangle $\hat{\Theta}$ in $\mathcal{C}$. Then $F_M(\hat{\Theta})$ is a distinguished n-triangle in $M\text{–}\mathrm{Mod}_{\mathcal{C}}$ since $G_M F_M$ is exact. The base of $F_M(\hat{\Theta})$ admits an idempotent endomorphism

$$\left(\xi_{a_{0,1}}\epsilon_{a_{0,1}}\,,\, \xi_{a_{0,2}}\epsilon_{a_{0,2}}\,,\, \ldots,\, \xi_{a_{0,n}}\epsilon_{a_{0,n}}\right).$$

The direct summand of the base of $F_M(\hat{\Theta})$ corresponding to this idempotent is simply the string $a_{0,1} \xrightarrow{f_{0,1}} \cdots \xrightarrow{f_{0,n-1}} a_{0,n}$ that we want to complete into an n-triangle.

By the already proven (TC n.3) – this is the key point of our approach – we can extend the above idempotent endomorphism of the base into an endomorphism of the whole n-triangle $F_M(\hat{\Theta})$ in M–$\mathrm{Mod}_{\mathcal{C}}$:

$$d : F_M(\hat{\Theta}) \longrightarrow F_M(\hat{\Theta}) .$$

Now, by Remark 5.15, every object in $F_M(\hat{\Theta})$ outside of the base is the cone of a morphism of the base, as in (5.4); hence we can apply Lemma 1.6 strictly speaking (not some extension to higher triangles). By part (c) of that Lemma 1.6, if we let $e := 3d^2 - 2d^3$, we have that $e = e^2$ is an idempotent on every object of $F_M(\hat{\Theta})$. By Remark 2.6, M–$\mathrm{Mod}_{\mathcal{C}}$ is idempotent-complete. So, by Remark 1.9, $F_M(\hat{\Theta})$ splits up as the direct sum of two n-triangles Θ and Θ' in M–$\mathrm{Mod}_{\mathcal{C}}$, such that Θ has the wanted base. It remains to see that Θ is distinguished. Simply apply G_M to the relation $F_M(\hat{\Theta}) = \Theta \oplus \Theta'$ and use that distinguished n-triangles in $\mathcal{C}$ are stable by direct summand, see Remark 5.16. □

5.18. **Corollary.** *Let $\mathcal{C}$ be a tensor triangulated category of order N (more precisely, we only need that, for every fixed object $A \in \mathcal{C}$, the functor $A \otimes -$ is exact up to order N). Let A be a separable ring object in $\mathcal{C}$ (Def. 3.1). Then the category of left A-modules in $\mathcal{C}$ has a unique triangulation of order N such that an n-triangle of A-modules for $n \leq N$ is distinguished exactly when the underlying n-triangle of objects of $\mathcal{C}$ is distinguished.* □

5.19. *Remark.* We can apply our results to the opposite category, in order to obtain the same statements for co-modules over co-ring objects, or more generally over co-monads. In short, a *comonad on* $\mathcal{C}$ is a triple (W, ∇, ϵ) where $W : \mathcal{C} \to \mathcal{C}$ is a functor (e.g. $W = H \otimes -$ for an coring object H), with comultiplication $\nabla : W \to W^2$ and counit $\epsilon : W \to \mathbb{1}$ making the dual of diagrams (2.2) commute. The category W–$\mathrm{Comod}_{\mathcal{C}}$ of W-comodules in $\mathcal{C}$ and morphisms thereof is defined as usual, which amounts to $W\text{–}\mathrm{Comod}_{\mathcal{C}} = \big(W^{\mathrm{op}}\text{–}\mathrm{Mod}_{\mathcal{C}^{\mathrm{op}}}\big)^{\mathrm{op}}$ where $W^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is the dual monad. There is a cofree functor $\mathcal{C} \to W$–$\mathrm{Comod}_{\mathcal{C}}$, $x \mapsto \big(W(x), \nabla_x\big)$, which is *right* adjoint to the forgetful functor W–$\mathrm{Comod}_{\mathcal{C}} \to \mathcal{C}$. Rafael's Theorem says that the forgetful functor is separable as a functor (Def. 3.7) if and only if W is a separable comonad, meaning that $\nabla : W \to W^2$ has a retraction as bi-comodule. When $\mathcal{C}$ is moreover idempotent-complete and triangulated of order $N \geq 2$ and when W is exact up to that order, then every W-comodule is a direct summand of a cofree one and the category W–$\mathrm{Comod}_{\mathcal{C}}$ admits a unique triangulation of same order N, characterized by the property that both functors $W\text{–}\mathrm{Comod}_{\mathcal{C}} \leftrightarrows \mathcal{C}$ are exact. This applies in particular to comodules over coring objects. (Phew!)

6. Examples

6.1. *Remark.* Following up on the first paragraph of the Introduction, if the ring object (or the monad) in our triangulated category descends to a separable ring object in some stable model and if the modules in that model themselves form a stable model category with a Quillen adjunction of the type "free-module/forgetful", then the derived adjunction between homotopy categories has good chances of satisfying the assumptions of Theorem 5.17 (d). Hence, by that result, the homotopy category of the modules would be equivalent to the modules in the homotopy category. So, we recover the same triangulated category as the one obtained via models, if the latter exists. Theorem 6.5 below provides an illustration of this phenomenon.

6.2. *Example.* Let (M_i, μ_i), $i = 1, 2$, be two additive monads on an additive category $\mathcal{C}$. Then, we can form a monad $M_1 \oplus M_2$ with component-wise multiplication

$$(M_1 \oplus M_2)^2 = M_1^2 \oplus M_1M_2 \oplus M_2M_1 \oplus M_2^2 \xrightarrow{\left(\begin{smallmatrix}\mu_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu_2\end{smallmatrix}\right)} M_1 \oplus M_2$$

and obvious unit. If M_1 and M_2 are separable then so is $M_1 \oplus M_2$.

In particular, starting with the trivial separable monad $\mathrm{Id}_{\mathcal{C}}$, we obtain by induction a collection of monads $\mathrm{Id}_{\mathcal{C}}^{\oplus n}$ for every $n \geq 0$. When $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ is monoidal, this corresponds to the ring object $\mathbb{1}^{\oplus n}$ with component-wise multiplication. It is easy to verify that the category of $\mathrm{Id}_{\mathcal{C}}^{\oplus n}$-modules is just $\mathcal{C} \times \cdots \times \mathcal{C}$, with n copies of $\mathcal{C}$. The free module functor maps x to $(x, \ldots, x)$ diagonally, whereas the forgetful functor G adds up all components $(x_1, \ldots, x_n) \mapsto x_1 \oplus \cdots \oplus x_n$. Note that G is not full and that there exists no retraction of G as a functor. The retraction H on morphisms, as in Definition 3.7, takes an $(n \times n)$-matrix to its diagonal. The triangulation on $\mathcal{C} \times \cdots \times \mathcal{C}$ is the obvious one, "component-wise".

6.3. *Example.* Let $\mathcal{C}$ be triangulated and let (L, λ) be a *Bousfield localization*, i.e. $L : \mathcal{C} \to \mathcal{C}$ is an exact functor and $\lambda : \mathrm{Id}_{\mathcal{C}} \to L$ is a natural transformation such that $L\lambda$ is an isomorphism $L \overset{\sim}{\to} L^2$ and $L\lambda = \lambda L$. Then the inverse of $L\lambda$, say $\mu : L^2 \to L$, defines a monad structure on L with unit $\eta = \lambda$. This monad is stably separable with $\sigma = \lambda L$ in Definition 3.5. The L-modules are just L-local objects, i.e. objects of $\mathrm{Ker}(L)^{\perp}$, see [14, § 4.9]. The Eilenberg-Moore adjunction coincides with $L : \mathcal{C} \leftrightarrows \mathrm{Ker}(L)^{\perp} : G$, where G is the *fully* faithful inclusion; see Remark 4.2.

6.4. *Example.* Let R be a commutative ring and $\mathcal{C} = \mathrm{D}(R\text{–Mod})$. Denote by $X[0]$ the complex with the module X in degree zero and zero elsewhere. The triangulated category $\mathcal{C}$ admits the usual derived tensor product $\otimes = \otimes_R^{\mathrm{L}}$ with unit $\mathbb{1} = R[0]$. A classical R-algebra A, i.e. in $R\text{–Mod}$, might not define a ring object in $\mathcal{C}$ because $(A \otimes_R A)[0] \neq A[0] \otimes A[0]$ unless A is *R-flat*.

6.5. **Theorem.** *Let R be a commutative ring and A be a flat and separable R-algebra. Then, A defines a ring object $A[0]$ in $\mathrm{D}(R\text{–Mod})$ and the category of $A[0]$-modules in $\mathrm{D}(R\text{–Mod})$ is canonically equivalent, as an ∞-triangulated category, to the derived category $\mathrm{D}(A\text{–Mod})$ of the ring A.*

Proof. We have an adjunction of *exact* functors $F : R\text{–Mod} \leftrightarrows A\text{–Mod} : G$, where $F(-) = A \otimes_R -$ and G is the direct image (forgetful) functor. Let η and ϵ be the unit and counit of this adjunction. Since F and G are exact, we get a derived adjunction that we denote $\mathrm{D}F : \mathrm{D}(R\text{–Mod}) \leftrightarrows \mathrm{D}(A\text{–Mod}) : \mathrm{D}G$, with unit and counit $\mathrm{D}\eta$ and $\mathrm{D}\epsilon$. The functors $\mathrm{D}F$ and $\mathrm{D}G$ are simply obtained on complexes by applying F and G in each degree. Similarly, the natural transformations $\mathrm{D}\eta$ and $\mathrm{D}\epsilon$ are just η and ϵ in each degree. Note that this derived adjunction induces the monad $M(-) = A[0] \otimes -$. Since A is separable in $R\text{–Mod}$, there exists a section $\xi : \mathrm{Id}_{A\text{–Mod}} \longrightarrow FG$ of the unit $\epsilon : FG \longrightarrow \mathrm{Id}_{A\text{–Mod}}$ by Proposition 3.11 and Remark 3.9. This section extends to a section $\mathrm{D}\xi : \mathrm{Id}_{\mathrm{D}(A\text{–Mod})} \longrightarrow \mathrm{D}F \circ \mathrm{D}G$ of the counit $\mathrm{D}\epsilon$, again simply defined on complexes as ξ in each degree. This proves that $\mathrm{D}G$ is stably separable. We can now conclude by Theorem 5.17 (d). $\square$

Recall that one way to define an *étale* commutative R-algebra S is to require S to be separable, flat and of finite presentation. See [13, Def. p. 104] or Grothendieck [10]. Hence the above result specializes to:

6.6. **Corollary.** *Let R be a commutative ring and let S be a commutative étale R-algebra. Then the category of $S[0]$-modules in* $\mathrm{D}(R\text{-Mod})$ *is canonically equivalent, as an ∞-triangulated category, to the derived category* $\mathrm{D}(S\text{-Mod})$. $\square$

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References

[1] A. Baker and B. Richter. Galois extensions of Lubin-Tate spectra. *Homology, Homotopy Appl.*, 10(3):27–43, 2008.

[2] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[3] P. Balmer. Tensor triangular geometry. In *International Congress of Mathematicians, Hyderabad (2010), Vol. II*, pages 85–112. Hindustan Book Agency, 2010.

[4] P. Balmer and M. Schlichting. Idempotent completion of triangulated categories. *J. Algebra*, 236(2):819–834, 2001.

[5] A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces (Luminy, 1981)*, Astérisque 100. SMF, Paris, 1982.

[6] G. Böhm, T. Brzeziński, and R. Wisbauer. Monads and comonads on module categories. *J. Algebra*, 322(5):1719–1747, 2009.

[7] A. Bruguières and A. Virelizier. Hopf monads. *Adv. Math.*, 215(2):679–733, 2007.

[8] F. DeMeyer and E. Ingraham. *Separable algebras over commutative rings*. Lecture Notes in Mathematics, Vol. 181. Springer-Verlag, Berlin, 1971.

[9] S. Eilenberg and J. C. Moore. Foundations of relative homological algebra. *Mem. Amer. Math. Soc. No.*, 55:39, 1965.

[10] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (20, 24, 28, 32), 1967.

[11] K. Hess. A general framework for homotopic descent and codescent. Preprint, 65 pages, available online at `arXiv:1001.1556v2`, 2010.

[12] H. Kleisli. Every standard construction is induced by a pair of adjoint functors. *Proc. Amer. Math. Soc.*, 16:544–546, 1965.

[13] M.-A. Knus and M. Ojanguren. *Théorie de la descente et algèbres d'Azumaya*. Lecture Notes in Mathematics, Vol. 389. Springer-Verlag, Berlin, 1974.

[14] H. Krause. Localization for triangulated categories. In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2010.

[15] M. Künzer. On derived categories. Diploma thesis, Universität Stuttgart, 1996.

[16] M. Künzer. Heller triangulated categories. *Homology, Homotopy Appl.*, 9(2):233–320, 2007.

[17] M. Künzer. Nonisomorphic Verdier octahedra on the same base. *J. Homotopy Relat. Struct.*, 4(1):7–38, 2009.

[18] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[19] G. Maltsiniotis. Catégories triangulées supérieures. Preprint available at `http://people.math.jussieu.fr/~maltsin/ps/triansup.ps`, 2006.

[20] C. Năstăsescu, M. Van den Bergh, and F. Van Oystaeyen. Separable functors applied to graded rings. *J. Algebra*, 123(2):397–413, 1989.

[21] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, 2001.

[22] M. D. Rafael. Separable functors revisited. *Comm. Algebra*, 18(5):1445–1459, 1990.

[23] J. Rognes. Galois extensions of structured ring spectra. stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):viii+137, 2008.

[24] J.-L. Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, 239, 1996.

[25] F. Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, 1985.

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