SEPARABLE COMMUTATIVE RINGS IN THE STABLE MODULE CATEGORY OF CYCLIC GROUPS

PAUL BALMER AND JON F. CARLSON

ABSTRACT. We prove that the only separable commutative ring-objects in the stable module category of a finite cyclic p-group G are the ones corresponding to subgroups of G. We also describe the tensor-closure of the Kelly radical of the module category and of the stable module category of any finite group.

CONTENTS

INTRODUCTION

Since 1960 and the work of Auslander and Goldman [AG60], an algebra A over a commutative ring R is called *separable* if A is projective as an $A \otimes_R A^{\mathrm{op}}$-module. This notion turns out to be remarkably important in many other contexts, where the module category $\mathcal{C} = R\text{-}\mathrm{Mod}$ and its tensor $\otimes = \otimes_R$ are replaced by an arbitrary tensor category $(\mathcal{C}, \otimes)$. A ring-object A in such a category $\mathcal{C}$ is *separable* if multiplication $\mu : A \otimes A \to A$ admits a section $\sigma : A \to A \otimes A$ as an A-A-bimodule in $\mathcal{C}$. See details in Section 1. Our main result (Theorem 4.1) concerns itself with modular representation theory of finite groups:

Main Theorem. *Let $\Bbbk$ be a separably closed field of characteristic $p > 0$ and let G be a cyclic p-group. Let A be a commutative and separable ring-object in the stable category $\Bbbk G\text{-}\mathrm{stmod}$ of finitely generated $\Bbbk G$-modules modulo projectives. Then there exist subgroups $H_1, \ldots, H_r \le G$ and an isomorphism of ring-objects $A \simeq \Bbbk(G/H_1) \times \cdots \times \Bbbk(G/H_r)$. (The ring structure on the latter is recalled below.)*

Separable and commutative ring-objects are particularly interesting in tensor-triangulated categories, like the above stable module category $\Bbbk G\text{-}\mathrm{stmod}$. There

Date: July 1, 2017.
2010 *Mathematics Subject Classification.* 20C20; 14F20, 18E30.
Key words and phrases. Separable, etale, ring-object, stable category.
First author was supported by NSF grant DMS-1600032 and Research Award of the Humboldt Foundation. The second author was partially supported by NSA grant H98230-15-1-0007 and by Simons Foundation grant 054813-01.

are several reasons for this. First, from the theoretical perspective, if $\mathcal{K}$ is a tensor-triangulated category (called *tt-category* for short) and if A is a separable and commutative ring-object in $\mathcal{K}$ (called *tt-ring* for short) then the category $A\text{-}\mathrm{Mod}_{\mathcal{K}}$, of A-modules in $\mathcal{K}$, remains a tt-category. See details in [Bal11]. On the other hand, from the perspective of applications, tt-rings actually come up in many examples. Let us remind the reader.

In algebraic geometry, given an *étale* morphism $f : Y \to X$ of noetherian and separated schemes, the object $A = \mathrm{R}f_*(\mathcal{O}_Y)$ is a tt-ring in $\mathrm{D}(X) = \mathrm{D}(\mathrm{Qcoh}(X))$, the derived category of X. Moreover, the category of A-modules in $\mathrm{D}(X)$ is equivalent to the derived category of Y, as a tt-category. This result is proved in [Bal16]. Shortly thereafter, and it is an additional motivation for the present paper, Neeman proved that these ring-objects $\mathrm{R}f_*(\mathcal{O}_Y)$, together with obvious localizations, are the only tt-rings in the derived category $\mathrm{D}(X)$. The precise statement is Theorem 7.10 in [Nee15]. In colloquial terms, the only tt-rings which appear in algebraic geometry come from the étale topology.

In view of the above, one might ask: What is the analogue of the "étale topology" in modular representation theory? This investigation was started in [Bal15]. Let $\Bbbk$ be a field and G a finite group, and consider X a finite G-set. Then the permutation $\Bbbk G$-module $A = \Bbbk X$ admits a multiplication $\mu : A \otimes A \to A$ defined by $\Bbbk$-linearly extending the rule $\mu(x\otimes x) = x$ and $\mu(x\otimes x') = 0$ for all $x \neq x'$ in X; its unit $\Bbbk \to \Bbbk X$ maps 1 to $\sum_{x\in X} x$. This commutative ring-object $A = \Bbbk X$ in $\Bbbk G\text{-}\mathrm{mod}$ is separable (use $\sigma(x) = x \otimes x$), and consequently gives a commutative separable ring-object in any tensor category which receives $\Bbbk G\text{-}\mathrm{mod}$ via a tensor functor. Hence, we inherit tt-rings $\Bbbk X$ in the derived category $\mathrm{D^b}(\Bbbk G\text{-}\mathrm{mod})$ and in the stable module category of $\Bbbk G$, which is both the additive quotient $\Bbbk G\text{-}\mathrm{stmod} = \Bbbk G\text{-}\mathrm{mod}/\Bbbk G\text{-}\mathrm{proj}$ and the Verdier (triangulated) quotient $\mathrm{D^b}(\Bbbk G\text{-}\mathrm{mod})/\,\mathrm{K^b}(\Bbbk G\text{-}\mathrm{proj})$.

Since finite G-sets are disjoint unions of G-orbits and since $\Bbbk(X \sqcup Y) \simeq \Bbbk X \times \Bbbk Y$ as rings, we can focus attention on tt-rings associated to subgroups $H \leq G$ as

$$A^G_H := \Bbbk(G/H)\,.$$

Here is an interesting fact established in [Bal15] about this tt-ring A^G_H. Let us denote by $\mathcal{K}(G)$ either the bounded derived category $\mathcal{K}(G) = \mathrm{D^b}(\Bbbk G\text{-}\mathrm{mod})$, or the stable category $\mathcal{K}(G) = \Bbbk G\text{-}\mathrm{stmod}$, or any variation removing the "boundedness" or "finite dimensionality" conditions. Then the category of A^G_H-modules in $\mathcal{K}(G)$ is equivalent as a tt-category to the corresponding category $\mathcal{K}(H)$ for the subgroup H:

$$A^G_H\text{-}\mathrm{Mod}_{\mathcal{K}(G)} \simeq \mathcal{K}(H)\,.$$

This description of restriction to a subgroup $\mathcal{K}(G) \to \mathcal{K}(H)$ as an 'étale extension' in the tt-sense is not specific to linear representation theory but holds in a variety of equivariant settings, from topology to C*-algebras, as shown in [BDS15].

We hope the above short survey motivates the reader for the study of tt-rings, and we now focus mostly on the stable category $\mathcal{K}(G) = \Bbbk G\text{-}\mathrm{stmod}$. In [Bal15, Question 4.7], the first author asked whether the above examples are the only ones:

Question. Let $\Bbbk$ be a separably closed field and G a finite group. Let A be a tt-ring (*i.e.* separable and commutative) in the stable category $\Bbbk G\text{-}\mathrm{stmod}$. Is there a finite G-set X such that $A \simeq \Bbbk X$ in $\Bbbk G\text{-}\mathrm{stmod}$?

Equivalently, one might ask: Given a tt-ring A in $\Bbbk G\text{-}\mathrm{stmod}$ which is *indecomposable* as a ring, must we have that $A \simeq \Bbbk(G/H)$ for some subgroup $H \leq G$?

Less formally, this is asking whether “the étale topology in modular representation theory” is completely determined by the subgroups of G, or whether some exotic tt-rings can appear. Our Main Theorem solves this problem for cyclic p-groups.

Some comments are in order. First, the reason to assume $\Bbbk$ separably closed is obvious: If $L/\Bbbk$ is a finite separable field extension, then one can consider L as a trivial $\Bbbk G$-module, and it surely defines a tt-ring in $\Bbbk G\text{-}\mathrm{stmod}$ that is indecomposable as a ring but that has really very little to do with the group G itself. Similarly, we focus on the finite-dimensional $\Bbbk G$-modules, to avoid dealing with (right) Rickard idempotents as explained in Remark 1.4.

We point out that the answer to the above Question is positive if $\Bbbk G\text{-}\mathrm{stmod}$ is replaced by the abelian category of $\Bbbk G$-modules (see [Bal15, Rem. 4.6]). If $\mathcal{C}$ is the category of $\Bbbk$-vector spaces over a field $\Bbbk$, then the only commutative and separable $A \in \mathcal{C}$ are the finite products $L_1 \times \cdots \times L_n$ of finite separable field extensions $L_1, \ldots, L_n$ of $\Bbbk$. See [DI71, § II.2] or [Nee15, § 1]. In particular, if we assume $\Bbbk$ separably closed, this ring is simply $\Bbbk \times \cdots \times \Bbbk$. Remembering the action of G on the corresponding set of idempotents is how the result is proved for $\Bbbk G\text{-}\mathrm{Mod}$ in [Bal15, Rem. 4.6]. For the derived category $\mathrm{D}(\Bbbk G\text{-}\mathrm{Mod})$ consider the following related argument. Under the monoidal functor $\mathrm{Res}^G_1 : \mathrm{D}(\Bbbk G\text{-}\mathrm{Mod}) \to \mathrm{D}(\Bbbk)$, any tt-ring A in $\mathrm{D}(\Bbbk G\text{-}\mathrm{Mod})$ must go to an object concentrated in degree 0, by the field case (see Neeman [Nee15, Prop. 1.6]). Hence, A has only homology in degree zero and belongs to the image of the fully faithful tensor functor $\Bbbk G\text{-}\mathrm{Mod} \hookrightarrow \mathrm{D}(\Bbbk G\text{-}\mathrm{Mod})$. We are therefore reduced to the module case and the same statement holds for $\mathrm{D}(\Bbbk G\text{-}\mathrm{Mod})$ as for $\Bbbk G\text{-}\mathrm{Mod}$: Their only commutative and separable rings are the announced $\Bbbk X$ for finite G-sets X.

The question for the stable category is much trickier, mostly because the “fiber” functor to the non-equivariant case, $\mathrm{Res}^G_1 : \Bbbk G\text{-}\mathrm{stmod} \to \Bbbk\text{-}\mathrm{stmod} = 0$, is useless.

Our treatment starts with the case of $G = C_p$, cyclic of prime order. This turns out to be the critical case. We then proceed relatively easily to C_{p^n} by induction on n. Only the case of C_4 requires an extra argument.

The reader might wonder how the result can be so difficult for such a “simple” category as $\Bbbk C_{p^n}\text{-}\mathrm{stmod}$. Let us point to the fact that for the arguably even simpler, non-equivariant category $\mathcal{C} = \Bbbk\text{-}\mathrm{Mod}$ of $\Bbbk$-vector spaces, the proof requires a couple of pages in DeMeyer-Ingraham [DI71, § II.2]. The alternate proof of Neeman [Nee15, § 1] is equally long. Our result relies on these predecessors. Most importantly, the tensor product in $\Bbbk C_{p^n}\text{-}\mathrm{stmod}$ becomes rather complicated, even for indecomposable modules. See Formula (2.27) for C_p itself. A critical new ingredient in the stable category of C_p is the fact that the symmetric module $S^{p-1}[i]$ over the indecomposable $\Bbbk C_p$-module $[i]$ of dimension i is projective, for every $i > 1$. This fact was established by Almkvist and Fossum in [AF78]. In addition, the Kelly radical of $\Bbbk C_p\text{-}\mathrm{stmod}$ is a tensor-ideal, a fact which we show in Section 2. It is a very special feature of this case, as we also explain: When p^2 divides the order of G, the Kelly radical of $\Bbbk G\text{-}\mathrm{stmod}$ is not a $\otimes$-ideal. More generally, in Section 2 we characterize completely the smallest $\otimes$-ideal containing the Kelly radical, for any finite group G. This is Theorem 2.20 which is of independent interest.

The question discussed here is related to the Galois group of the stable module category as an ∞-category, as discussed by Mathew [Mat16, § 9], although neither result seems to imply the other.

We remind the reader that for a finite group G and a field $\Bbbk$ of characteristic $p > 0$, the stable category $\Bbbk G\text{-}\operatorname{stmod}$ is the category whose objects are finitely generated $\Bbbk G$-modules and whose morphism are given by $\operatorname{Hom}_{\Bbbk G\text{-}\operatorname{stmod}}(M,N) = \operatorname{Hom}_{\Bbbk G}(M,N)/\operatorname{PHom}_{\Bbbk G}(M,N)$ where PHom indicates those homomorphisms that factor through a projective module.

Acknowledgements: Both authors would like to thank Bielefeld University for its support and kind hospitality during a visit when this work was undertaken. We are also thankful to Danny Krashen, Akhil Mathew and Greg Stevenson for valuable discussions.

1. Separable ring-objects

In this section, we review the needed fundamental results on separable ring-objects in tensor categories, not necessarily triangulated at first.

Assume that $\mathcal{C}$ is a *tensor category*, meaning an additive, symmetric monoidal category such that $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is additive in each variable. We denote by $\mathbb{1}$ the $\otimes$-unit. A *ring-object* A in $\mathcal{C}$ is a triple (A, μ, u) where $A \in \operatorname{Obj}(\mathcal{C})$, $\mu : A \otimes A \to A$ is an associative multiplication, $\mu(\mu \otimes A) = \mu(A \otimes \mu)$, and the morphism $u : \mathbb{1} \to A$ is a two-sided unit, $\mu(A \otimes u) = 1_A = \mu(u \otimes A)$. (If $\mathcal{C}$ were not additive, a common terminology would be "monoid" instead of "ring-object".) The ring-object A is *commutative* if $\mu(12) = \mu$, where $(12) : A \otimes A \overset{\sim}{\to} A \otimes A$ is the swap of factors. By associativity, the composite of multiplications $A^{\otimes n} \xrightarrow{\mu} A^{\otimes n-1} \to \cdots \to A^{\otimes 2} \xrightarrow{\mu} A$ does not depend on the bracketing and we simply denote it by $\mu : A^{\otimes n} \to A$.

In this setting, an *A-module in the tensor category* $\mathcal{C}$ is a pair (M, ρ) where M is an object of the given category $\mathcal{C}$ (not some 'external' abelian group) and $\rho : A \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying the usual axioms of associativity and unital action. Such modules and their A-linear morphisms form an additive category $A\text{-}\operatorname{Mod}_{\mathcal{C}}$. It comes with the so-called *Eilenberg-Moore* adjunction

$$F_A : \mathcal{C} \rightleftarrows A\text{-}\operatorname{Mod}_{\mathcal{C}} : U_A$$

where $F_A(X) = (A \otimes X, \mu \otimes X)$ is the free A-module and its right adjoint $U_A(M, \rho) = M$ is the functor forgetting the action. This material is classical, and is recalled with more details in [Bal11, § 2] for instance.

1.1. *Definition.* A ring-object A as above is *separable* if there exists $\sigma : A \to A \otimes A$ such that $\mu\sigma = 1_A$ and $\sigma\mu = (\mu \otimes A)(A \otimes \sigma) = (A \otimes \mu)(\sigma \otimes A)$. This amounts to saying that A is projective as an $A \otimes A^{\mathrm{op}}$-module.

1.2. *Example.* As in the Introduction, for a subgroup $H \le G$, the separable commutative rings $A^G_H := \Bbbk(G/H)$ in $\mathcal{C} = \Bbbk G\text{-}\operatorname{stmod}$ has multiplication $\mu : A^G_H \otimes A^G_H \to A^G_H$ extending $\Bbbk$-linearly the formulas $\mu(x \otimes x) = x$ and $\mu(x \otimes x') = 0$ for all $x \neq x' \in G/H$, and unit $u : \Bbbk \to A^G_H$ given by $u(1) = \sum_{x \in G/H} x$. The multiplication μ is split by the map $\sigma : A^G_H \to A^G_H \otimes A^G_H$, that takes $x \in G/H$ to $\sigma(x) = x \otimes x$.

1.3. **Proposition.** *Let A be a separable commutative ring-object in a tensor category $\mathcal{C}$. Then we have:*

(a) ***Relative semisimplicity of** A **over** $\mathcal{C}$: Let $f : M' \to M$ and $g : M \to M''$ be two morphisms of A-modules in $\mathcal{C}$ such that the underlying sequence of objects $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is split-exact in $\mathcal{C}$. Then the sequence is split-exact* *as a sequence of A-modules, i.e. f admits an A-linear retraction $r : M \to M'$ such that $\left(\begin{smallmatrix} r \\ g \end{smallmatrix}\right) : M \overset{\sim}{\to} M' \oplus M''$ is an isomorphism of A-modules.*

(b) ***No nilpotence**: Suppose that $A = I \oplus J$ in $\mathcal{C}$ and that I is an ideal (i.e. the morphism $A \otimes I \to A \otimes A \overset{\mu}{\to} A$ factors via $I \hookrightarrow A$). Suppose that I is nilpotent (i.e. there exists $n \geq 1$ such that $I^{\otimes n} \to A^{\otimes n} \xrightarrow{\mu} A$ is zero). Then $I = 0$.*

Proof. For (a), consider a retraction $r : M \to M'$ of $f : M' \to M$ in $\mathcal{C}$, so that $r\,f = 1_{M'}$. Now, let $\bar{r} : M \to M'$ be the following composite:

$$M \xrightarrow{u \otimes 1} A \otimes M \xrightarrow{\sigma \otimes 1} A \otimes A \otimes M \xrightarrow{1 \otimes \rho} A \otimes M \xrightarrow{1 \otimes r} A \otimes M' \xrightarrow{\rho'} M'.$$

This morphism is still a retraction of f but is now A-linear. The reader unfamiliar with separability could check these facts to appreciate the non-triviality of this property. Indeed, the above construction $r \mapsto \bar{r}$ yields in general a well-defined map $H : \mathrm{Hom}_{\mathcal{C}}(M, M') \to \mathrm{Hom}_{A\text{-}\mathrm{Mod}_{\mathcal{C}}}(M, M')$ which retracts the inclusion $\mathrm{Hom}_{A\text{-}\mathrm{Mod}_{\mathcal{C}}}(M, M') \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(M, M')$ and which is natural in M and M' in the sense that $H(f r f') = f\,H(r) f'$ whenever f and f' are A-linear. See [BBW09, 2.9 (1)] or [BV07, Prop. 6.3] for details.

Part (b) follows easily from (a), since now we have that $A = I \oplus J$ as A-modules, that is, as ideals. Consider the unit morphism $u : \mathbb{1} \to A = I \oplus J$. The composition

$$\mathbb{1} = \mathbb{1}^{\otimes n} \xrightarrow{u^{\otimes n}} A^{\otimes n} = (I \oplus J)^{\otimes n} \xrightarrow{\mu} A$$

is equal to u itself. Since $(I \oplus J)^{\otimes n} = I^{\otimes n} \oplus (J \otimes (\ldots))$, since I is nilpotent and since J is an ideal, the above composition factors via $J \hookrightarrow A$ for n big enough. So the ideal $J \subseteq A$ contains the unit. This readily implies $J = A$ and $I = 0$ as claimed. $\square$

1.4. *Remark.* In general there are examples of separable commutative ring-objects in the big stable category $\Bbbk G\text{-}\mathrm{StMod}$ for a finite group G, that differ from the objects associated to finite G-sets as in the Introduction. These arise, for instance, as Rickard idempotents [Ric97]. Recall briefly, that to any specialization-closed subset Y in the spectrum $V_G(\Bbbk) = \mathrm{Proj}(\mathrm{H}^*(G, \Bbbk))$ of homogeneous prime ideals in the cohomology ring of G, we associate an exact triangle in $\Bbbk G\text{-}\mathrm{StMod}$

$$\mathcal{E}_Y \xrightarrow{\gamma} \Bbbk \xrightarrow{\lambda} \mathcal{F}_Y \longrightarrow$$

where $\mathcal{E}_Y \otimes \mathcal{F}_Y = 0$ and where $\mathcal{E}_Y$ belongs to the localizing subcategory generated by $\mathcal{C}_Y := \left\{ M \in \Bbbk G\text{-}\mathrm{stmod} \,\middle|\, V_G(M) \subseteq Y \right\}$ and $\mathcal{F}_Y$ to its orthogonal, *i.e.* $\mathrm{Hom}_{\Bbbk G\text{-}\mathrm{StMod}}(M, \mathcal{F}_Y) = 0$ for all $M \in \mathcal{C}_Y$. These properties uniquely characterize $\mathcal{E}_Y$ and $\mathcal{F}_Y$. Then there is a multiplication $\mu : \mathcal{F}_Y \otimes \mathcal{F}_Y \to \mathcal{F}_Y$ inverse to the isomorphism $\lambda \otimes \mathcal{F}_Y = \mathcal{F}_Y \otimes \lambda$, turning $\mathcal{F}_Y$ into a tt-ring in $\Bbbk G\text{-}\mathrm{StMod}$. The $\Bbbk G$-module $\mathcal{F}_Y$ is not finitely generated as soon as Y is non-empty and proper.

This phenomenon is a special case of the general observation that a right Rickard idempotent in any tensor-triangulated category is a tt-ring. Its category of modules is nothing but the corresponding Bousfield (smashing) localization.

In the proof of our main theorem, we come across the following tensor category. Let us describe its separable commutative ring-objects.

1.5. **Proposition.** *Let $\Bbbk$ be a field of characteristic 2. The only commutative separable ring in the $\otimes$-category of $\mathbb{Z}/2$-graded $\Bbbk$-vector spaces are concentrated in degree zero (i.e. the separable $\Bbbk$-algebras with trivial grading).*

Proof. As extension-of-scalars from $\Bbbk$ to any bigger field $L/\Bbbk$ is faithful, we can assume that $\Bbbk$ is separably closed. The functor which maps a $\mathbb{Z}/2$-graded $\Bbbk$-vector space (V_0, V_1) to the 'underlying' $\Bbbk$-vector space $V_0 \oplus V_1$ is a tensor functor. Suppose $A = (V_0, V_1)$ is a commutative separable $\mathbb{Z}/2$-graded $\Bbbk$-algebra and let us prove that $V_1 = 0$. Since $\Bbbk$ is separably closed, the underlying $\Bbbk$-algebra of A is trivial. Let $\varphi : \Bbbk \times \cdots \times \Bbbk \xrightarrow{\sim} V_0 \oplus V_1$ be an isomorphism of ungraded $\Bbbk$-algebras. Consider $e = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Bbbk^{\times n}$ one of the idempotents, and let $\varphi(e) = v_0 + v_1$ with $v_0 \in V_0$ and $v_1 \in V_1$ in A. The relation $\varphi(e)^2 = \varphi(e)$, commutativity and characteristic two, give $v_0 = v_0^2 + v_1^2$ and $v_1 = 2v_0v_1 = 0$. So $v_0 \in V_0$ is an idempotent. Hence, V_0 contains n orthogonal idempotents, showing that $\dim_\Bbbk V_0 \geq n = \dim_\Bbbk(A)$. Hence $A = V_0$ and $V_1 = 0$ as claimed. □

2. The Kelly radical and the tensor

Throughout this section, $\Bbbk$ is a field of positive characteristic p dividing the order of G and modules over a group algebra $\Bbbk G$ are assumed to be finite dimensional. We begin by recalling the definition of Kelly radical of a category [Kel64].

2.1. *Definition.* The *radical* of an additive category $\mathcal{C}$ is the ideal of morphisms

$$\mathrm{Rad}_{\mathcal{C}}(M, N) = \big\{ f : M \to N \,\big|\, \text{ for all } g : N \to M,\ 1_M - gf \text{ is invertible} \big\}.$$

When $M = N$, the ideal $\mathrm{Rad}_{\mathcal{C}}(M) := \mathrm{Rad}_{\mathcal{C}}(M, M)$ is the Jacobson radical of the ring $\mathrm{End}_{\mathcal{C}}(M)$.

In this section, we give a characterization of the tensor-closure of the Kelly radical, both in the module category $\Bbbk G\text{-}\mathrm{mod}$ and in the stable category $\Bbbk G\text{-}\mathrm{stmod}$. In particular, we show that if G is a cyclic p-group, then the Kelly radical is a tensor ideal. The results of this section are far stronger than what is needed for later sections, but they are of independent interest.

2.2. *Remark.* Recall that in an additive category $\mathcal{C}$ an *ideal of morphisms* $\mathcal{I}$ consists of a collection of subgroups $\mathcal{I}(M, N) \subseteq \mathrm{Hom}_{\mathcal{C}}(M, N)$ for all object M, N (we only consider *additive* ideals in this paper), which is closed under composition:

$$\mathrm{Hom}(N, N') \circ \mathcal{I}(M, N) \circ \mathrm{Hom}(M', M) \subseteq \mathcal{I}(M', N')\,. \tag{2.3}$$

Then for any decompositions $M \simeq M_1 \oplus \cdots \oplus M_m$ and $N \simeq N_1 \oplus \cdots \oplus N_n$ a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ belongs to $\mathcal{I}(M, N)$ if and only if each $f_{ji} = \mathrm{pr}_j \circ f \circ \mathrm{inj}_i$ belongs to $\mathcal{I}(M_i, N_j)$, where $\mathrm{inj}_i : M_i \rightarrowtail M$ and $\mathrm{pr}_j : N \twoheadrightarrow N_j$ are the given injections and projections. Hence, an ideal $\mathcal{I}$ of morphisms in a Krull-Schmidt category $\mathcal{C}$ is determined by the subgroups $\mathcal{I}(M, N) \subseteq \mathrm{Hom}_{\mathcal{C}}(M, N)$ for indecomposable M, N. Conversely, a collection of such subgroups $\mathcal{I}(M, N) \subseteq \mathrm{Hom}_{\mathcal{C}}(M, N)$ for all indecomposable M, N defines a unique ideal $\mathcal{I}$ if (2.3) is satisfied for all M, M', N, N' indecomposable.

For any ideal $\mathcal{I}$, we can form the additive quotient category $\mathcal{C}/\mathcal{I}$

$$Q : \mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{I} \tag{2.4}$$

which has the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{\mathcal{C}}(M, N)/\mathcal{I}(M, N)$. When $\mathcal{I} = \mathrm{Rad}_{\mathcal{C}}$, we have $1_M \notin \mathrm{Rad}_{\mathcal{C}}(M)$ unless $M = 0$. The corresponding functor $Q : \mathcal{C} \twoheadrightarrow \mathcal{C}/\mathrm{Rad}_{\mathcal{C}}$ is conservative (detects isomorphisms).

2.5. *Remark.* When $\mathcal{C}$ is a tensor category, an ideal $\mathcal{I}$ of morphisms is called a *tensor ideal* (abbreviated $\otimes$-ideal) if $f\otimes g \in \mathcal{I}$ whenever $f \in \mathcal{I}$. This is equivalent to asking only $f \otimes L \in \mathcal{I}(M \otimes L, N \otimes L)$ for every $f \in \mathcal{I}(M, N)$ and every object L. In that case, $\mathcal{C}/\mathcal{I}$ becomes a $\otimes$-category and the quotient $Q : \mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{I}$ is a $\otimes$-functor.

It should be emphasized that the definition of the Kelly radical $\mathrm{Rad}_{\mathcal{C}}$ is not related to the existence of a tensor structure on $\mathcal{C}$. In particular, for any specific $\otimes$-category $\mathcal{C}$, the ideal $\mathrm{Rad}_{\mathcal{C}}$ may or may not be a $\otimes$-ideal. So the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathrm{Rad}_{\mathcal{C}}$ is not necessarily a $\otimes$-functor, even if in specific cases $\mathcal{C}/\mathrm{Rad}_{\mathcal{C}}$ admits some 'natural' tensor structure for independent reasons.

2.6. *Definition.* We denote by $\mathrm{Rad}^{\otimes}$ the smallest $\otimes$-ideal containing Rad, *i.e.* the $\otimes$-ideal it generates. We call $\mathrm{Rad}^{\otimes}$ the *tensor-closure of the Kelly radical.*

Our discussion of the tensor-closure $\mathrm{Rad}^{\otimes}$ passes through the algebraic closure $\overline{\mathbb{k}}$ of $\mathbb{k}$. For this reason, we isolate some well known facts as a preparation:

2.7. **Proposition.** *Let $\overline{\mathbb{k}}$ be an algebraic closure of $\mathbb{k}$. Let M and N be finite dimensional $\mathbb{k}G$-modules, and consider the $\overline{\mathbb{k}}G$-modules $\overline{\mathbb{k}}\otimes_{\mathbb{k}} M$ and $\overline{\mathbb{k}}\otimes_{\mathbb{k}} N$. Then:*

(a) *There is a canonical and natural isomorphism*

$$\mathrm{Hom}_{\overline{\mathbb{k}}G}(\overline{\mathbb{k}}\otimes_{\mathbb{k}} M, \overline{\mathbb{k}}\otimes_{\mathbb{k}} N) \simeq \overline{\mathbb{k}}\otimes_{\mathbb{k}} \mathrm{Hom}_{\mathbb{k}G}(M,N)\,. \tag{2.8}$$

(b) *Under (2.8) for $M = N$, we have that $\overline{\mathbb{k}}\otimes_{\mathbb{k}} \mathrm{Rad}_{\mathbb{k}G}(M) \subseteq \mathrm{Rad}_{\overline{\mathbb{k}}G}(\overline{\mathbb{k}}\otimes_{\mathbb{k}} M)$.*

(c) *Suppose M and N are indecomposable. Then $\overline{\mathbb{k}}\otimes_{\mathbb{k}} M$ and $\overline{\mathbb{k}}\otimes_{\mathbb{k}} N$ have a nonzero direct summand in common if and only if $M \simeq N$.*

(d) *Suppose that the trivial module $\overline{\mathbb{k}}$ is a direct summand of the $\overline{\mathbb{k}}G$-module $\overline{\mathbb{k}}\otimes_{\mathbb{k}} M$. Then $\mathbb{k}$ is a direct summand of M.*

Proof. The canonical morphism $\overline{\mathbb{k}}\otimes \mathrm{Hom}_{\mathbb{k}G}(M,N) \longrightarrow \mathrm{Hom}_{\overline{\mathbb{k}}G}(\overline{\mathbb{k}}\otimes M, \overline{\mathbb{k}}\otimes N)$, between left exact functors in M (for N fixed) is an isomorphism when $M = \mathbb{k}G$, hence also for every finitely presented $\mathbb{k}G$-module M. This gives (2.8). For (b), it suffices to observe that $\overline{\mathbb{k}}\otimes_{\mathbb{k}} \mathrm{Rad}_{\mathbb{k}G}(M)$ is a nilpotent two-sided ideal of the ring $\overline{\mathbb{k}}\otimes_{\mathbb{k}} \mathrm{End}_{\mathbb{k}G}(M)$, which is isomorphic to the ring $\mathrm{End}_{\overline{\mathbb{k}}G}(\overline{\mathbb{k}}\otimes_{\mathbb{k}} M)$ by (a). See [Lam91, Thm. 5.14] if necessary. For (c), assume that $M \not\simeq N$ and suppose that U is a direct summand of both $\overline{\mathbb{k}}\otimes M$ and $\overline{\mathbb{k}}\otimes N$. Then there exist homomorphisms $f : \overline{\mathbb{k}}\otimes M \to \overline{\mathbb{k}}\otimes N$ and $g : \overline{\mathbb{k}}\otimes N \to \overline{\mathbb{k}}\otimes M$ such that gf is an idempotent endomorphism of $\overline{\mathbb{k}}\otimes M$ with image isomorphic to U. By (2.8), $f = \sum_{i=1}^{m} a_i \otimes f_i$ and $g = \sum_{j=1}^{n} b_j \otimes g_j$ for some $a_i, b_j \in \overline{\mathbb{k}}$ and $f_i : M \to N$ and $g_j : N \to M$. As $M \not\simeq N$, all compositions $g_j f_i : M \to M$ belong to the radical since they factor through N. Because M is finite-dimensional, the radical of $\mathrm{Hom}_{\mathbb{k}G}(M,M)$ is nilpotent. So there exists an integer ℓ such that $(gf)^{\ell} = 0$. But gf is idempotent, so $gf = 0$ and therefore $U \simeq \mathrm{im}(gf) = 0$. For (d), we can assume M indecomposable. Then (d) follows from (c) with $N = \mathbb{k}$. $\square$

2.9. *Remark.* Another tool in our discussion of $\mathrm{Rad}^{\otimes}$ is rigidity. Recall that a tensor category $\mathcal{C}$ is *rigid* if there exists a 'dual' $(-)^{\vee} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that every $M \in \mathcal{C}$ induces an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[d, bend right=50, "M\otimes-"'] \\ \mathcal{C} \arrow[u, bend right=50, "M^{\vee}\otimes-"'] \end{tikzcd}$$

This holds for instance for $\mathcal{C} = \Bbbk G\text{-}\mathrm{mod}$ or for $\mathcal{C} = \Bbbk G\text{-}\mathrm{stmod}$ with $M^\vee = \mathrm{Hom}_\Bbbk(M, \Bbbk)$ with the usual G-module structure $(g \cdot f)(m) = f(g^{-1}m)$. The above adjunction comes with a unit $\eta_M : \mathbb{1} \to M^\vee \otimes M$ and a counit $\epsilon_M : M \otimes M^\vee \to \mathbb{1}$, which in our example are respectively the $\Bbbk$-linear map $\Bbbk \to M^\vee \otimes M \simeq \mathrm{End}_\Bbbk(M)$ mapping $1 \in \Bbbk$ to the identity of M, and the $\Bbbk$-linear map given by the swap of factors followed by the trace: $M \otimes M^\vee \simeq M^\vee \otimes M \simeq \mathrm{End}_\Bbbk(M) \xrightarrow{\mathrm{tr}} \Bbbk$.

Rigidity allows us to isolate a critical property of a module, which is at the heart of the distinction between Rad and $\mathrm{Rad}^\otimes$.

2.10. *Definition.* A finitely generated $\Bbbk G$-module M is said to be *$\otimes$-faithful* provided the functor $M \otimes - : \Bbbk G\text{-}\mathrm{stmod} \to \Bbbk G\text{-}\mathrm{stmod}$ is faithful.

2.11. *Remark.* We use the stable category, not the ordinary category, for the above simple definition. In $\Bbbk G\text{-}\mathrm{mod}$, every non-zero M induces a faithful functor $M \otimes -$. We are nevertheless going to give several equivalent formulations in $\Bbbk G\text{-}\mathrm{mod}$. Note that a projective $\Bbbk G$-module P is never $\otimes$-faithful, since $P = 0$ in $\Bbbk G\text{-}\mathrm{stmod}$.

2.12. *Example.* A $\Bbbk G$-module M such that $\dim(M)$ is prime to $p = \mathrm{char}(\Bbbk)$ is $\otimes$-faithful since $\eta_M : \mathbb{1} \to M^\vee \otimes M$ is split by $\dim(M)^{-1} \cdot \mathrm{tr} : M^\vee \otimes M \to \Bbbk$. A converse holds when $\Bbbk$ is algebraically closed, as we recall in Theorem 2.14 below.

2.13. **Proposition.** *Let M be a finitely generated $\Bbbk G$-module. The following properties are equivalent:*

(i) *The $\Bbbk G$-module M is $\otimes$-faithful (Definition 2.10).*

(ii) *Some indecomposable summand of M is $\otimes$-faithful.*

(iii) *There exists a finitely generated $\Bbbk G$-module X such that $X \otimes M$ is $\otimes$-faithful.*

(iv) *The unit $\eta : \Bbbk \to M^\vee \otimes M$ is a split monomorphism of $\Bbbk G$-modules.*

(v) *The unit $\eta : \Bbbk \to M^\vee \otimes M$ is a split monomorphism in the stable category $\Bbbk G\text{-}\mathrm{stmod}$.*

(vi) *The trace $\mathrm{tr} : M^\vee \otimes M \to \Bbbk$ (or equivalently the counit $\epsilon_M : M \otimes M^\vee \to \Bbbk$) is a split epimorphism of $\Bbbk G$-modules, or equivalently in the stable category.*

(vii) *$\Bbbk$ is a direct summand of $M^\vee \otimes M$ in $\Bbbk G\text{-}\mathrm{mod}$, or equivalently in $\Bbbk G\text{-}\mathrm{stmod}$.*

(viii) *$\Bbbk$ is a direct summand of $X \otimes M$ for some finitely generated $\Bbbk G$-module X.*

If $\overline{\Bbbk}$ is an algebraic closure of $\Bbbk$, then the above are further equivalent to:

(ix) *The $\overline{\Bbbk}G$-module $\overline{\Bbbk} \otimes_\Bbbk M$ is $\otimes$-faithful.*

(x) *Some direct summand of $\overline{\Bbbk} \otimes_\Bbbk M$ has dimension that is not divisible by p.*

Proof. Recall that $\Bbbk G\text{-}\mathrm{mod}$ and $\Bbbk G\text{-}\mathrm{stmod}$ are Krull-Schmidt categories and that M has the same indecomposable summands in both, except for the projectives which vanish stably. Hence, the two formulations of (vii) are indeed equivalent (and (viii) is unambiguous). It is straightforward to check (i) $\iff$ (ii) $\iff$ (iii) from Definition 2.10. Also obvious are (iv)$\Rightarrow$(v)$\Rightarrow$(vii)$\Rightarrow$(viii)$\Rightarrow$(i).

Let us show that (i)$\Rightarrow$(iv). As $\mathcal{C} = \Bbbk G\text{-}\mathrm{stmod}$ is a rigid tensor-*triangulated* category, we can choose an exact triangle $N \xrightarrow{\xi} \mathbb{1} \xrightarrow{\eta} M^\vee \otimes M \to \Sigma N$ in $\mathcal{C}$. The unit-counit relation shows that $M \otimes \eta$ is a split monomorphism. Hence, $M \otimes \xi = 0$ in $\mathcal{C}$, and $\xi = 0$ since $M \otimes - : \mathcal{C} \to \mathcal{C}$ is assumed faithful. Consequently, η is a split monomorphism, by a standard property of triangulated categories, see [Nee01, Cor. 1.2.7].

Similarly, (vi)⇒(i) is trivial and (i)⇒(vi) is proven as above.

At this stage, we know that (i)–(viii) are all equivalent. Now, property (vii) holds for M over $\Bbbk$ if and only if it holds for $\bar{\Bbbk} \otimes_{\Bbbk} M$ over $\bar{\Bbbk}$ by Proposition 2.7 (d). Hence, (i)–(viii) are also equivalent to (ix). We already saw that (x)⇒(ix) in Example 2.12. The converse, (ix)⇒(x), holds by the following more general theorem of Dave Benson and the second author (applied in the case $\Bbbk = \bar{\Bbbk}$). □

2.14. **Theorem** ([BC86]). *Suppose that M and N are absolutely indecomposable $\Bbbk G$-modules (i.e. remain indecomposable over the algebraic closure) and suppose that the trivial module $\Bbbk$ is a direct summand of $M \otimes N$. Then $\dim(M)$ is not divisible by p, $N \simeq M^\vee$, the multiplicity of $\Bbbk$ as direct summand of $M \otimes N$ is one, the unit $\eta_M : \Bbbk \to M^\vee \otimes M$ (mapping 1 to 1_M) is a split monomorphism and the trace $\mathrm{tr} : M^\vee \otimes M \to \Bbbk$ is a split epimorphism.*

2.15. *Remark.* The reason for the assumption of absolute indecomposability is illustrated in an easy example. Let $G = \langle x \rangle \simeq C_3$, be a cyclic group of order 3, and $\Bbbk = \mathbb{F}_2$, the prime field with two elements. The algebra $\Bbbk G$ is a semisimple and as a module over itself it decomposes $\Bbbk G \simeq \Bbbk \oplus M$, where M has dimension 2. If a cube root of unity ζ is adjoined to $\Bbbk$, then M splits as a sum of two one-dimensional modules on which x acts by multiplication by ζ on one and by ζ^2 on the other. Then it is not difficult to see that $M \otimes M \simeq \Bbbk \oplus \Bbbk \oplus M$, since x has an eigenspace, with eigenvalue one, of dimension 2 on the tensor product. Of course, in this example, the characteristic of the field does not divide the order of the group. Another example can be constructed by inflating this module to $\Bbbk G$ where G is the alternating group A_4, along the map $G \to C_3$ whose kernel is the Sylow 2-subgroup. More complicated examples also exist.

The next lemma is a corollary of the multiplicity-one property in Theorem 2.14.

2.16. **Lemma.** *Assume that $\Bbbk = \bar{\Bbbk}$ is algebraically closed. Let M be an indecomposable $\Bbbk G$-module of dimension prime to p. Let $j : \Bbbk \rightarrowtail M^\vee \otimes M$ be any split monomorphism and $q : M^\vee \otimes M \twoheadrightarrow \Bbbk$ any split epimorphism. Then:*

(a) *The composite $q \circ j : \Bbbk \to \Bbbk$ is not the zero map.*

(b) *Let $f : M \to M$ be any morphism. Then the composite $q(1 \otimes f)j : \Bbbk \to M^\vee \otimes M \xrightarrow{1 \otimes f} M^\vee \otimes M \to \Bbbk$ is a non-zero multiple of the trace of f.*

Proof. Since $M^\vee \otimes M \simeq \Bbbk \oplus L$ where L contains no $\Bbbk$ summand, the split morphisms j and q must be respectively a non-zero multiple of the (canonical) split morphisms $\eta : \Bbbk \rightarrowtail M^\vee \otimes M$ and $\mathrm{tr} : M^\vee \otimes M \twoheadrightarrow \Bbbk$, plus morphisms factoring through L, which are in particular in the Kelly radical. Computing the composite in (b), using that $\mathrm{Rad}(\Bbbk) = 0$, we see that $q(1 \otimes f)j$ is a non-zero multiple of

$$\Bbbk \xrightarrow{\eta} M^\vee \otimes M \xrightarrow{1 \otimes f} M^\vee \otimes M \xrightarrow{\mathrm{tr}} \Bbbk$$

which is the trace of f. Part (a) follows from (b) for $f = 1_M$. □

Let us return to our discussion of the tensor-closure $\mathrm{Rad}^{\otimes}$ of the radical. The relevance of ⊗-faithfulness (Definition 2.10) for this question is isolated in the following result. Recall that $p = \mathrm{char}(\Bbbk)$ divides $|G|$.

2.17. **Proposition.** *Let M be a finitely generated $\Bbbk G$-module which is* not *⊗-faithful. Then the identity 1_M of M belongs to the tensor-closure $\mathrm{Rad}^{\otimes}_{\mathcal{C}}$ of the* *radical, both in* $\mathcal{C} = \Bbbk G\text{-}\operatorname{mod}$ *and in* $\mathcal{C} = \Bbbk G\text{-}\operatorname{stmod}$*. Hence,* $\operatorname{Rad}^{\otimes}_{\mathcal{C}}(M,M) = \operatorname{Hom}_{\mathcal{C}}(M,M)$*.*

In particular, the Kelly radical of $\mathcal{C} = \Bbbk G\text{-}\operatorname{mod}$ *is never a* $\otimes$*-ideal. Moreover, if there exists such an* M *not projective, then the Kelly radical is not* $\otimes$*-ideal in the stable category* $\mathcal{C} = \Bbbk G\text{-}\operatorname{stmod}$*.*

Proof. By assumption, the unit $\eta_M : \mathbb{1} \to M^\vee \otimes M$ is not a split monomorphism. This means that η_M belongs to the radical, as $\operatorname{End}_{\mathcal{C}}(\mathbb{1}) \simeq \Bbbk$. Hence, the morphism $M \otimes \eta_M : M \to M \otimes M^\vee \otimes M$ belongs to $\operatorname{Rad}^{\otimes}_{\mathcal{C}}$. By the unit-counit relation, we have that $1_M = (\epsilon_M \otimes M) \circ (M \otimes \eta_M)$. Hence, 1_M belongs to the ideal of morphisms $\operatorname{Rad}^{\otimes}_{\mathcal{C}}$ as claimed. This phenomenon readily implies that the Kelly radical is not a $\otimes$-ideal if $M \neq 0$ in the category $\mathcal{C}$, since the identity of a non-zero object never belongs to $\operatorname{Rad}_{\mathcal{C}}$. In the ordinary category $\mathcal{C} = \Bbbk G\text{-}\operatorname{mod}$, the free module $M = \Bbbk G$ gives an example of such a non-zero M. On the other hand, we need M to be not projective in order to have that $M \neq 0$ in the stable category $\mathcal{C} = \Bbbk G\text{-}\operatorname{stmod}$. □

We are therefore naturally led to consider the following ideal of morphisms, first in $\Bbbk G\text{-}\operatorname{mod}$ and later in $\Bbbk G\text{-}\operatorname{stmod}$ (Definition 2.23).

2.18. *Definition.* For M and N indecomposable $\Bbbk G$-modules, let $\mathcal{I}(M,N)$ be the subspace of $\operatorname{Hom}_{kG}(M,N)$ defined as follows.

(1) If $M \not\simeq N$, then $\mathcal{I}(M,N) := \operatorname{Rad}(\operatorname{Hom}_{kG}(M,N)) = \operatorname{Hom}_{kG}(M,N)$.
(2) If M is not $\otimes$-faithful, then $\mathcal{I}(M,M) := \operatorname{Hom}_{kG}(M,M)$.
(3) If M is $\otimes$-faithful, then $\mathcal{I}(M,M) := \operatorname{Rad}(\operatorname{Hom}_{kG}(M,M))$.

(When $N \simeq M$, we define $\mathcal{I}(M,N)$ via (2) or (3) transported by any such isomorphism.) This collection of $\mathcal{I}(M,N)$ is closed under composition, as discussed in Remark 2.2. Hence, it defines a unique ideal of morphisms in $\Bbbk G\text{-}\operatorname{mod}$, still denoted $\mathcal{I}$. It clearly contains the radical, from which it only differs in case (2).

As earlier, let us see that this ideal is stable under algebraic field extensions.

2.19. **Lemma.** *Let* $\mathcal{I}_{\Bbbk}$ *denote the ideal defined in Definition 2.18 for* $\Bbbk G$*-modules. Let* $\overline{\Bbbk}$ *be an algebraic closure of* $\Bbbk$*. Suppose that* M *and* N *are* $\Bbbk G$*-modules. A map* $f : M \to N$ *belongs to* $\mathcal{I}_{\Bbbk}(M,N)$ *if and only if* $\overline{\Bbbk} \otimes f$ *belongs to* $\mathcal{I}_{\overline{\Bbbk}}(\overline{\Bbbk} \otimes M, \overline{\Bbbk} \otimes N)$*.*

Proof. We may assume that M and N are indecomposable. If $M \not\simeq N$, then no nonzero summand of $\overline{\Bbbk} \otimes M$ is isomorphic to any summand of $\overline{\Bbbk} \otimes N$ by Proposition 2.7 (c). Then $\mathcal{I}_{\Bbbk}(M,N) = \operatorname{Hom}_{\Bbbk G}(M,N)$ and similarly for $\overline{\Bbbk}$, so there is nothing to prove about f. Suppose therefore that $M \simeq N$. Recall from Proposition 2.13 that M is $\otimes$-faithful if and only if $\overline{\Bbbk} \otimes M$ is. So, if M is not $\otimes$-faithful, then neither is any summand of $\overline{\Bbbk} \otimes M$ and again $\mathcal{I}_{\Bbbk}(M,M)$ and $\mathcal{I}_{\overline{\Bbbk}}(\overline{\Bbbk} \otimes M, \overline{\Bbbk} \otimes M)$ are the entire groups of homomorphisms and there is nothing to prove about f.

Let us then assume M $\otimes$-faithful and take $f : M \to M$. If f does not belong to $\mathcal{I}_{\Bbbk}(M,M) = \operatorname{Rad}_{\Bbbk G}(M)$ then f is an isomorphism and then so is $\overline{\Bbbk} \otimes f$. As one summand of $\overline{\Bbbk} \otimes M$ is $\otimes$-faithful, the isomorphism $\overline{\Bbbk} \otimes f$ does not belong to $\mathcal{I}_{\overline{\Bbbk}}(\overline{\Bbbk} \otimes M, \overline{\Bbbk} \otimes M)$. Conversely, suppose that $f : M \to M$ belongs to $\mathcal{I}_{\Bbbk}(M,M)$, which is the radical of $\operatorname{End}_{\Bbbk G}(M)$. By Proposition 2.7 (b), $\overline{\Bbbk} \otimes f$ belongs to the radical, which is contained in the larger ideal $\mathcal{I}_{\overline{\Bbbk}}$. □

2.20. **Theorem.** *The ideal* $\mathcal{I}$ *of Definition 2.18 is the tensor-closure* $\operatorname{Rad}^{\otimes}$ *of the Kelly radical (Definition 2.6) of the category* $\Bbbk G\text{-}\operatorname{mod}$*.*

The critical point is the following:

2.21. **Lemma.** *Assume that $\Bbbk = \overline{\Bbbk}$ is algebraically closed. Let M and N be indecomposable $\Bbbk G$-modules of dimension prime to p and let $f : M \to N$ be a homomorphism. Suppose that X is a $\Bbbk G$-module such that there is a common indecomposable summand $U \leq M \otimes X$ and $U \leq N \otimes X$ of dimension prime to p, with split injection $i : U \rightarrowtail M \otimes X$ and split projection $p : N \otimes X \twoheadrightarrow U$. Suppose that $p \circ (f \otimes X) \circ i : U \to U$ is an isomorphism. Then $f : M \to N$ is an isomorphism.*

Proof. Let $g := p \circ (f \otimes 1_X) \circ i : U \xrightarrow{\sim} U$ be our isomorphism. Tensoring with $U^\vee$, we have an automorphism $g \otimes 1$ of $U \otimes U^\vee$. The composite

$$\begin{tikzcd} \Bbbk \arrow[r, tail, "\eta_U"] & U \otimes U^\vee \arrow[r, "i\otimes 1"] \arrow[rrr, bend right, "g\otimes 1"', "\simeq"] & M \otimes X \otimes U^\vee \arrow[r, "f\otimes 1\otimes 1"] & N \otimes X \otimes U^\vee \arrow[r, "p\otimes 1"] & U \otimes U^\vee \arrow[r, two heads, "\mathrm{tr}_U"] & \Bbbk \end{tikzcd}$$

is non-zero by Lemma 2.16 (a), that is, an isomorphism. Decomposing $X \otimes U^\vee$ into a sum of indecomposable summands V, the above isomorphism $\Bbbk \to \Bbbk$ is the sum of the corresponding compositions

$$\Bbbk \longrightarrow M \otimes V \xrightarrow{f\otimes 1} N \otimes V \longrightarrow \Bbbk$$

over all these V. This holds because the middle map $f \otimes 1 \otimes 1$ above “is” the identity on the $X \otimes U^\vee$ factor. Since the sum of these morphisms is non-zero, one of them must be non-zero, *i.e.* an isomorphism, for some V. Applying Theorem 2.14 to $M \otimes V$ and again to $N \otimes V$, we have that $V \simeq M^\vee$ and that $V \simeq N^\vee$. In particular, $M \simeq N$. Replacing N by M using such an isomorphism, we can assume that $f : M \to M$ is an endomorphism. The above isomorphism $\Bbbk \to \Bbbk$ now becomes

$$\Bbbk \overset{j}{\rightarrowtail} M \otimes M^\vee \xrightarrow{f\otimes 1} M \otimes M^\vee \overset{q}{\twoheadrightarrow} \Bbbk$$

for some morphisms j and q which must be a split mono and a split epi respectively, since that composite is an isomorphism. By Lemma 2.16 (b) this composite is also a non-zero multiple of the trace of f. Because the composite is an isomorphism, $\mathrm{tr}(f) \neq 0$, and therefore f cannot be nilpotent. It follows that f cannot belong to the radical of the finite-dimensional $\Bbbk$-algebra $\mathrm{End}_{\Bbbk G}(M)$. Hence, f is invertible, as claimed. □

Proof of Theorem 2.20. We already know that the Kelly radical is contained in $\mathcal{I}$. The first thing to note is that $\mathcal{I}$ is contained in the tensor ideal $\mathrm{Rad}^\otimes$ generated by the radical. This follows from the definition of $\mathcal{I}$ and Proposition 2.17.

It remains to show that $\mathcal{I}$ is a tensor ideal. To begin, assume that $\Bbbk = \overline{\Bbbk}$ is algebraically closed. In this case an indecomposable $\Bbbk G$-module is $\otimes$-faithful if and only if its dimension is not divisible by p. Let $f \in \mathcal{I}(M, N)$ with M and N indecomposable, and let X be an object. We want to show that $f \otimes X$ belongs to $\mathcal{I}$. To test this, we need to decompose $M \otimes X$ and $N \otimes X$ into a sum of indecomposable. Suppose ab absurdo that $f \otimes X$ does not belong to $\mathcal{I}(M \otimes X, N \otimes X)$. Since $\mathcal{I}(-,-)$ is often equal to the whole of $\mathrm{Hom}(-,-)$, the only way that $f \otimes X$ cannot belong to $\mathcal{I}$ is that $M \otimes X$ and $N \otimes X$ admit a common direct summand U, of dimension prime to p, on which $f \otimes X$ is invertible (see case (3) of Definition 2.18). So U is $\otimes$-faithful, hence so are $M \otimes X$ and $N \otimes X$, and therefore M and N as well; see Proposition 2.13, (ii)⇒(i) and (iv)⇒(i). Therefore $\mathcal{I}(M, N) = \mathrm{Rad}(M, N)$. In summary, $f \in \mathcal{I}(M,N)$ is non-invertible but $f \otimes X$ is invertible on some common indecomposable summand U of $M \otimes X$ and $N \otimes X$, of dimension prime to p. This is exactly the situation excluded by Lemma 2.21 (since $\Bbbk$ is algebraically closed).

Now consider the case of a general field $\Bbbk$, perhaps not algebraically closed. Suppose that $f : M \to N$ is in $\mathcal{I}_{\Bbbk}(M,N)$. Let X be any $\Bbbk G$-module. Then $\overline{\Bbbk} \otimes (f \otimes X) = (\overline{\Bbbk} \otimes f) \otimes (\overline{\Bbbk} \otimes X)$ is in $\mathcal{I}_{\overline{\Bbbk}}(\overline{\Bbbk} \otimes (X \otimes M), \overline{\Bbbk} \otimes (X \otimes N))$. But now Lemma 2.19, implies that $f \otimes X$ is in $\mathcal{I}_{\Bbbk}(M \otimes X, N \otimes X)$. Thus $\mathcal{I}_{\Bbbk}$ is a tensor ideal and the proof is complete. ☐

2.22. *Remark.* Everything that we have done in the module category will translate directly to the stable category, except that the ideal needs to be defined somewhat differently. Recall that $\mathrm{PHom}_{\Bbbk G}(M,N)$ is the subspace of $\mathrm{Hom}_{\Bbbk G}(M,N)$ consisting of all homomorphisms from M to N that factor through a projective module. It is very easy to see that $\mathrm{PHom}_{\Bbbk G}(M,N) \subseteq \mathcal{I}(M,N)$. Indeed, the only case, for M and N indecomposable, that $\mathrm{PHom}_{\Bbbk G}(M,N)$ is not in the Kelly radical, occurs when $M \simeq N$ is projective, and no projective module is $\otimes$-faithful.

2.23. *Definition.* In the stable category, we define

$$\mathcal{I}_s(M,N) \;=\; \mathcal{I}(M,N)/\,\mathrm{PHom}_{\Bbbk G}(M,N)\,.$$

This clearly is an ideal in the stable category $\Bbbk G\text{-}\,\mathrm{stmod}$. Explicitly, from Definition 2.18, we have for M and N indecomposable in $\mathcal{C} := \Bbbk G\text{-}\,\mathrm{stmod}$:

(1) If $M \not\simeq N$, then $\mathcal{I}_s(M,N) := \mathrm{Rad}_{\mathcal{C}}(M,N) = \mathrm{Hom}_{\mathcal{C}}(M,N)$.
(2) If M is not $\otimes$-faithful, then $\mathcal{I}_s(M,M) := \mathrm{Hom}_{\mathcal{C}}(M,M)$.
(3) If M is $\otimes$-faithful, then $\mathcal{I}_s(M,M) := \mathrm{Rad}_{\mathcal{C}}(M,M)$.

2.24. **Theorem.** *The ideal $\mathcal{I}_s$ is the tensor-closure* $\mathrm{Rad}^{\otimes}$ *of the Kelly radical (Definition 2.6) of the category* $\Bbbk G\text{-}\,\mathrm{stmod}$.

Proof. By Theorem 2.20, $\mathcal{I}$ is a $\otimes$-ideal, and since $\mathrm{PHom}_{\Bbbk G}$ is also a $\otimes$-ideal, so is $\mathcal{I}_s = \mathcal{I}/\,\mathrm{PHom}_{\Bbbk G}$. The rest follows easily as before. Indeed, $\mathcal{I}_s$ clearly contains the radical, and agrees with it in most cases, except in the case of $\mathcal{I}_s(M,M)$ for M not $\otimes$-faithful where $\mathcal{I}_s(M,M) = \mathrm{Hom}_{\mathcal{C}}(M,M)$. But in that case, this is also $\mathrm{Rad}^{\otimes}_{\mathcal{C}}(M,M)$ by Proposition 2.17 for $\mathcal{C} = \Bbbk G\text{-}\,\mathrm{stmod}$. ☐

This leads directly to the following.

2.25. **Theorem.** *Let $\Bbbk$ be a field of characteristic $p > 0$ and let G be a finite group of order divisible by p. Then the Kelly radical $\mathrm{Rad}_{\mathcal{C}}$ of the category $\mathcal{C} = \Bbbk G\text{-}\,\mathrm{stmod}$ is a $\otimes$-ideal if and only if p^2 does not divide the order of G.*

Proof. First suppose that p^2 divides the order of G. Let Q be a subgroup of order p in G and let $M = \Bbbk_Q^{\uparrow G} = \Bbbk G \otimes_{\Bbbk Q} \Bbbk_Q$ where $\Bbbk_Q$ is the trivial $\Bbbk Q$-module. Let S be a Sylow p-subgroup of G that contains Q. By the Mackey Theorem, we have that

$$M_{\downarrow S} \quad\simeq\quad \bigoplus_{SxQ} \Bbbk^{\uparrow S}_{S \cap xQx^{-1}}$$

where the sum is over a collection of representatives of the S-Q double cosets in G. The modules $\Bbbk^{\uparrow S}_{S \cap xQx^{-1}}$ are absolutely indecomposable, have dimension divisible by p and are not projective if $S \cap xQx^{-1} \neq \{1\}$. Hence, some non-projective summand of M must fail to be $\otimes$-faithful, and the Kelly radical is not tensor closed by Proposition 2.17.

On the other hand, suppose that a Sylow p-subgroup S of G is cyclic of order p. We show that every non-projective indecomposable $\bar{\Bbbk}G$-module has dimension prime to p. This implies that every $\Bbbk G$-module is $\otimes$-faithful by Proposition 2.13 (x)$\Rightarrow$(i). So assume that $\Bbbk = \bar{\Bbbk}$. Let $S = \langle h \rangle$, and $t = h - 1$, so that $\Bbbk S = \Bbbk[t]/(t^p)$.

First consider the case that S is normal in G. Let M be an indecomposable $\Bbbk G$-modules. Then it is known that M is uniserial, meaning that the subsets $M_i = t^i M$ are $\Bbbk G$-submodule for $i = 0, \ldots, p-1$, and the quotients M_i/M_{i+1} are all irreducible and conjugate to one another. Moreover, because M is not projective, $M_{p-1} = \{0\}$. Thus the dimension of M is $r \cdot \dim(M/M_1)$ where r is the least integer such that $M_r = \{0\}$. The quotient M/M_1 is an irreducible $\Bbbk G/S$-module. Because $\Bbbk$ is algebraically closed, the dimension of M/M_1 divides $|G/S|$ and is prime to p (see [CR66] (33.7) which applies also in this case). Hence M has dimension prime to p.

If S is not normal in G, then let $N = N_G(S)$. Let M be a non-projective indecomposable $\Bbbk G$-module. Then M is a direct summand of $U^{\uparrow G}$ for U an indecomposable $\Bbbk N$-module that is a direct summand of the restriction $M_{\downarrow N}$. Note that U is not projective as otherwise M is also projective. Thus, by the previous case, the dimension of U is not divisible by p. By the Mackey Theorem

$$(U^{\uparrow G})_{\downarrow S} \;\simeq\; \bigoplus_{SxN} ((x \otimes U)_{\downarrow S \cap xNx^{-1}})^{\uparrow S}$$

where the sum is over a set of representatives of the S-N double cosets in G. But notice that $S \cap xNx^{-1} = \{1\}$ if $x \notin N$. Hence, $U^{\uparrow G}$ can have only one non-projective direct summand which must be M. All other direct summands must have dimension divisible by p. Because $\dim(U^{\uparrow G}) = |G : N| \;\dim(U)$, we have that p does not divide the dimension of M. $\square$

2.26. *Example.* Let $\mathcal{C} = \Bbbk C_{p^n}\text{-}\,\mathrm{stmod}$ the stable module category over a cyclic p-group $C_{p^n} = \langle\, g \,\big|\, g^{p^n} = 1 \,\rangle$. For $\Bbbk$ of characteristic p, we have a ring isomorphism $\Bbbk C_{p^n} \xrightarrow{\sim} \Bbbk[t]/t^{p^n}$ given by $g \mapsto t+1$. Every indecomposable module has the form

$$[i] := \Bbbk[t]/t^i$$

for $i = 1, \ldots, p^n$, the last one being projective. Hence, the indecomposable objects in the stable category $\mathcal{C}$ are the $[i]$ for $1 \le i \le p^n - 1$. The Kelly radical of $\mathcal{C}$ is generated by the morphisms $\alpha_i : [i] \to [i+1]$ given by multiplication by t and the morphisms $\beta_i : [i] \to [i-1]$ given by the projection. In particular, the radical of $\mathrm{End}_{\mathcal{C}}([i])$ is generated by the morphism $\beta_{i+1}\alpha_i : [i] \to [i]$ given by multiplication by t. The modules are all absolutely indecomposable so that none of this depends heavily on the field $\Bbbk$, as long as it has characteristic p, of course. Consequently, the Kelly radical is preserved under field extensions $\Bbbk C_{p^n}\text{-}\,\mathrm{stmod} \longrightarrow \Bbbk' C_{p^n}\text{-}\,\mathrm{stmod}$.

Consider the quotient $\mathcal{C} \overset{Q}{\twoheadrightarrow} \mathcal{D} := \mathcal{C}/\operatorname{Rad}_{\mathcal{C}}$ of (2.4). In this example, the category $\mathcal{D}$ consists simply of $p^n - 1$ copies of the category of $\Bbbk$-vector spaces, since we have $\mathrm{End}_{\mathcal{D}}(Q([i])) \simeq \Bbbk$ for all i and $\mathrm{Hom}_{\mathcal{D}}(Q([i]), Q([j])) = 0$ for $i \ne j$. There is a natural component-wise tensor on $\mathcal{D}$ in this example. However, this tensor on $\mathcal{D}$ never makes the quotient functor $Q : \mathcal{C} \to \mathcal{D}$ into a $\otimes$-functor, except for $G = C_2$ where Q is an isomorphism. For $n = 1$, we have seen that $\operatorname{Rad}_{\mathcal{C}}$ is a $\otimes$-ideal, hence there is *another* tensor structure on $\mathcal{D}$ which makes Q a $\otimes$-functor. For $n \ge 2$, the radical is simply not a $\otimes$-ideal. (See Theorem 2.25.)

In the case of $G = C_p$, the fact that the Kelly radical of the stable category is a tensor ideal can also be deduced from the tensor formula, for $i \le j$:

$$(2.27)\quad [i]\otimes[j] \simeq \begin{cases} [j-i+1]\oplus[j-i+3]\oplus\cdots\oplus[j+i-1] & \text{if } i+j\le p \\ [j-i+1]\oplus[j-i+3]\oplus\cdots\oplus[2p-i-j-1] & \text{if } i+j>p. \end{cases}$$

This formula is a consequence of a calculation of Premet [Pre91]. See [CFP08, Cor. 10.3] for details. Observe that all indecomposable summands $[k]$ of $[i]\otimes[j]$ have the same parity as $j-i+1$ (even for $i \ge j$ of course since $\otimes$ is symmetric). In particular, every morphism f between $[i]\otimes[j]$ and $[i\pm 1]\otimes[j]$ belongs to the radical since no summand of the source of f is isomorphic to any summand of its target. On the other hand, we saw that the radical is generated by the morphisms $\alpha_i : [i] \to [i+1]$ (multiplication by t) and $\beta_i : [i] \to [i-1]$ (projection). It follows that $\alpha_i \otimes [j]$ and $\beta_i \otimes [j]$ belong to the radical for all j.

3. The case of the group of prime order

Let p be a prime, C_p the cyclic group of order p and $\Bbbk$ a field of characteristic p.

3.1. **Theorem.** *Let $A \in \Bbbk C_p\text{-}\mathrm{StMod}$ be a tt-ring in the (big) stable module category. Then there exists finitely many finite separable field extensions $L_1, \ldots, L_n$ over $\Bbbk$ such that $A \simeq L_1 \times \cdots \times L_n$ as tt-rings in $\Bbbk C_p\text{-}\mathrm{StMod}$, where $L_1 \times \cdots \times L_n$ is equipped with trivial C_p-action.*

We need the following general preparations.

3.2. *Remark.* Let S be a finite group whose order is invertible in $\Bbbk$. Let M be a finite dimensional $\Bbbk S$-module. Suppose that $M^S = 0$, meaning M has no non-trivial S-fixed vector. Then there is also no nonzero kS-homomophism $M \to \Bbbk$, since such a map would split by semisimplicity of $\Bbbk S$, and thus $\Bbbk$ would be a direct summand of M. It follows that any $\Bbbk$-linear map $\nu : M \to M'$ such that $\nu(sm) = \nu(m)$ for all $m \in M$ and all $s \in S$ must be zero, since such a map ν has to factor through a trivial kS-module.

In the proof of Theorem 3.1, we use the above argument in a slightly more general setting, where M is an object of a category $\mathcal{D} = \oplus_{i=1}^{m}(\Bbbk\text{-}\mathrm{Mod})_i$ obtained by taking a finite (co)product of copies of the category $\Bbbk\text{-}\mathrm{Mod}$ (as additive categories). Since two copies of $\Bbbk\text{-}\mathrm{Mod}$ for different indices have no non-zero morphisms between them in $\mathcal{D}$, one easily reduces to the above case.

3.3. *Remark.* More generally, let $\mathcal{C}$ be a $\Bbbk$-linear idempotent-complete category, and let S be a finite group whose order is invertible in $\Bbbk$. Let M be an object of $\mathcal{C}$ on which S acts, in the sense that we have a group homomorphism $S \to \mathrm{Aut}_{\mathcal{C}}(M)$. We can then describe the S-fixed subobject M^S as an explicit direct summand of M, namely the summand corresponding to the idempotent endomorphism given by the image of the central idempotent $e = \frac{1}{|S|}\sum_{s\in S} s$ of $\Bbbk S$ in $\mathrm{End}_{\mathcal{C}}(M)$. So we have $M^S = e\cdot M = \ker(1-e)$ and $M = e\cdot M \oplus (1-e)\cdot M$. If now $F : \mathcal{C} \to \mathcal{D}$ is a $\Bbbk$-linear functor between such categories, it follows from the above description that $(F(M))^S = e\cdot F(M) \simeq F(e\cdot M) = F(M^S)$, as long as $F(M)$ is equipped with the obvious S-action $S \to \mathrm{Aut}_{\mathcal{C}}(M) \to \mathrm{Aut}_{\mathcal{D}}(F(M))$ induced by F.

3.4. *Remark.* The above ideas are applied below to the symmetric group $S = \mathfrak{S}_{p-1}$ on $p-1$ letters acting on an object M in a $\Bbbk$-linear category $\mathcal{C}$, where $p > 0$ is the characteristic of $\Bbbk$. The three $\Bbbk$-linear categories we use are in turn the module category $\Bbbk C_p\text{-}\mathrm{Mod}$, the stable category $\Bbbk C_p\text{-}\mathrm{StMod}$ and finally its quotient $\mathcal{D} = \Bbbk C_p\text{-}\mathrm{StMod}/\mathrm{Rad}$ by the Kelly radical. The object M with an action of $S = \mathfrak{S}_{p-1}$ is $M = [i]^{\otimes(p-1)}$ with action by permutation of the factors; and we also consider the images of M under the quotient functors $P : \Bbbk C_p\text{-}\mathrm{Mod} \to \Bbbk C_p\text{-}\mathrm{StMod}$ and $Q : \Bbbk C_p\text{-}\mathrm{StMod} \to \mathcal{D}$. Since both functors are quotient functors, *i.e.* only change the morphisms, the object in question remains the "same" $[i]^{\otimes(p-1)}$, if one wishes. By Remark 3.3, its S-fixed sub-object M^S is preserved by the functors P and Q.

Proof of Theorem 3.1. When $p = 2$, the category $\Bbbk C_2\text{-}\mathrm{StMod}$ is equivalent to $\Bbbk\text{-}\mathrm{Mod}$. That is, the only non-projective indecomposable module is the trivial module $\Bbbk$ and every module is stably isomorphic to a coproduct of trivial modules. See [CJ64] or [War69]. Thus, the theorem is trivially true in this case. As a consequence, we assume hereafter that p is odd.

Consider the additive quotient of $\Bbbk C_p\text{-}\mathrm{StMod}$ by its Kelly radical:

$$\Bbbk C_p\text{-}\mathrm{StMod} \xrightarrow{\;Q\;}\!\!\!\!\!\rightarrow \mathcal{D} := \frac{\Bbbk C_p\text{-}\mathrm{StMod}}{\mathrm{Rad}(\Bbbk C_p\text{-}\mathrm{StMod})}$$

By Theorem 2.25 (or Example 2.26), this Kelly radical is a tensor-ideal. This also uses the fact that every object of $\Bbbk C_p\text{-}\mathrm{StMod}$ is a coproduct of finite-dimensional ones, see [CJ64, War69] again. Therefore the above functor Q is a tensor functor. Hence, $B := Q(A)$ is a separable commutative ring in $\mathcal{D}$.

The quotient category $\mathcal{D}$ is actually abelian semisimple. Indeed, in the tt-category $\Bbbk C_p\text{-}\mathrm{StMod}$ every object is a (possibly infinite) coproduct of finite dimensional indecomposables and there are $p-1$ indecomposables up to isomorphism: $[1] = \Bbbk$, $[2], \ldots, [p-1]$; furthermore for all $i \neq j$, we have $\mathrm{Rad}([i],[j]) = \mathrm{Hom}([i],[j])$ and we have $\mathrm{Hom}([i],[i])/\mathrm{Rad}([i],[i]) \simeq \Bbbk$. This means that the quotient

$$\mathcal{D} \simeq \bigoplus_{i=1}^{p-1} (\Bbbk\text{-}\mathrm{Mod})_i$$

is a (co)product of copies of the category of $\Bbbk$-vector spaces, indexed by $i = 1, \ldots, p-1$. (Finite coproducts of additive categories coincide with their products.) The subtlety about the quotient category $\mathcal{D}$ comes from its tensor product, which is governed by Formula (2.27).

Now, choose $1 < i < p$ and suppose that A has a copy of $[i]$ among its direct summands (in $\Bbbk C_p\text{-}\mathrm{StMod}$). Consider the $\otimes$-power $M := [i]^{\otimes(p-1)}$ in $\Bbbk C_p\text{-}\mathrm{Mod}$, with the obvious action of the symmetric group $S = \mathfrak{S}_{p-1}$ on $p-1$ letters by permuting the factors as announced in Remark 3.4. Since $(p-1)!$ is invertible in $\Bbbk$, we can describe the fixed subobject M^S as in Remark 3.3. Indeed, $(p-1)! = -1$ in $\Bbbk$. So in other words, the symmetric power of $[i]$ equals $S^{p-1}[i] = M^S = e \cdot M$ where $e = -\sum_{s\in\mathfrak{S}_{p-1}} s$.

By Remark 3.4, we have $QP(M)^S \simeq QP(M^S)$ in $\mathcal{D}$. However, by the work of Almkvist and Fossum [AF78] we have that $M^S = S^{p-1}[i]$ is projective for $i \geq 2$, as we assume here. Hence, $P(M^S) \simeq 0$ in $\Bbbk C_p\text{-}\mathrm{StMod}$ and therefore $QP(M)^S = 0$ as well. We can then apply Remark 3.2 to the object $QP(M)$, which is really just $[i]^{\otimes(p-1)}$ but now viewed in $\mathcal{D}$. By that remark, the morphism $\nu : QP([i]^{\otimes p-1}) \to Q(A^{\otimes p-1}) \xrightarrow{Q\mu} Q(A)$ is zero, since it satisfies $\nu \circ s = \nu$ for all $s \in \mathfrak{S}_{p-1}$ by commutativity of μ.

In summary, we have shown that every direct summand $Q([i])$ of $B = Q(A)$ with $i > 1$ has to be nilpotent in the separable commutative ring-object B of $\mathcal{D}$. Let now $I \subseteq B = Q(A)$ be the ideal generated by all direct summands $Q([i]) \subseteq Q(A)$ for $i > 1$ in the semisimple abelian category $\mathcal{D}$. Because the category $\mathcal{D}$ is abelian and semisimple, this ideal I consists of the sum of the images $\mu(U \otimes V)$ where U is any summand of B and V is any direct summand of B that is isomorphic to $Q([i])$. By the above discussion, this ideal $I \subseteq B$ is nilpotent. Hence, it must be zero by Proposition 1.3 (b). This shows that A has no direct summand isomorphic to $[i]$ for $i > 1$.

Thus we have proved that A is a $\Bbbk C_p$-module with trivial C_p-action, and it belongs to the image of the fully faithful tensor functor $\pi^* : \Bbbk\text{-}\mathrm{Mod} \hookrightarrow \Bbbk C_p\text{-}\mathrm{StMod}$ (where $\pi : C_p \to 1$) and we reduce again to the field case. ☐

4. The case of the general cyclic group

Let $\Bbbk$ be a field of characteristic $p > 0$, and let C_{p^n} be the cyclic group of order p^n for $n \geq 1$. The following statement implies the theorem given in the Introduction:

4.1. **Theorem.** *Let $A \in \Bbbk C_{p^n}\text{-}\mathrm{stmod}$ be a tt-ring in the stable module category. Then there exists a commutative and separable ring-object A in $\Bbbk C_{p^n}\text{-}\mathrm{mod}$ whose image in the stable category is A. Explicitly, if we assume $\Bbbk$ separably closed, there exist a finite C_{p^n}-set X such that $A \simeq \Bbbk X$, or equivalently there exist subgroups $H_1, \ldots, H_r \leq G$ such that $A \simeq A^G_{H_1} \times \cdots \times A^G_{H_r}$ (see Example 1.2).*

We need a little preparation.

4.2. **Proposition.** *Suppose that $G = \langle g \rangle$ is a cyclic group of order $p^n > 1$ and $H = \langle g^{p^{n-m}} \rangle$ is the subgroup of order $p^m > 1$. Suppose that M is a $\Bbbk G$-module having no nonzero projective summands and suppose that the restriction $M_{\downarrow H}$ has a nonzero $\Bbbk H$-projective summand. Then $M_{\downarrow H}$ has an indecomposable summand of dimension $p^m - 1 = |H| - 1$.*

Proof. We may assume without loss of generality that M is indecomposable. Let r be the dimension of M. Let $t = g - 1$, so that $\Bbbk G \simeq \Bbbk[t]/(t^{p^n})$. Then M has a basis $v_1, \ldots, v_r$ such that $tv_i = v_{i+1}$ for all $i = 1, \ldots, r-1$ and $tv_r = 0$. For $i > r$, set by convention $v_i := 0$. The algebra of the subgroup H is $\Bbbk[y]/(y^{p^m})$ where $y = g^{p^{n-m}} - 1 = t^{p^{n-m}}$. As a $\Bbbk H$-module $M_{\downarrow H}$ is generated by $v_1, \ldots, v_{p^{n-m}-1}$, and $yv_i = v_{p^{n-m}+i}$. If $r < p^{n-m}$, then the $\Bbbk H$-module $M_{\downarrow H}$ would be trivial; hence $r \geq p^{n-m}$. It is then a straightforward exercise to show that

$$M_{\downarrow H} \;\simeq\; \Bbbk H v_1 \oplus \Bbbk H v_2 \oplus \cdots \oplus \Bbbk H v_{p^{n-m}}.$$

The fact that $M_{\downarrow H}$ has a projective direct summand means that $y^{p^m-1}M \neq \{0\}$ and therefore $r > (p^m - 1)p^{n-m} = p^n - p^{n-m}$. Because M is not projective $r < p^n$. Now write $r = (p^n - p^{n-m}) + s$ where $1 \leq s < p^{n-m}$. Then we have that $y^{p^m-1}v_{s+1} = v_{r+1} = 0$ and $y^{p^m-2}v_{s+1} \neq 0$. Thus the submodule $\Bbbk H v_{s+1}$ is an indecomposable direct summand of $M_{\downarrow H}$ having dimension $p^m - 1 = |H| - 1$ as asserted. ☐

This proposition is quite useful except in the fringe case where $p = 2$ and $n = 2$. This is the unique case in which $p^{n-1} - 1$ equals 1, and we handle it separately.

4.3. **Proposition.** *Let $A \in \Bbbk C_4\text{-}\mathrm{stmod}$ be a tt-ring over the cyclic group of order four, with $\mathrm{char}(\Bbbk) = 2$. Then A has no indecomposable summand $[3]$ of dimension 3.*

Proof. Use the Notation $[i]$ of Example 2.26 for $i = 1, 2, 3$. Note that $[1] = \mathbb{1}$ and $[3] \simeq \Sigma\mathbb{1}$, so that $[3] \otimes [3] = [1]$ in the stable category. This follows by the tensor formula for shifts. Direct inspection shows that $[2] \otimes [2] \simeq [2] \oplus [2]$. It follows that there is a well-defined $\otimes$-ideal $\mathcal{I}$ of morphisms in $\mathcal{C} = \Bbbk C_4\text{-}\mathrm{stmod}$ which consists of those morphism which factor via some $[2]^{\oplus m}$ for some $m \geq 1$. (One can verify that this is the ideal $\mathrm{Rad}^{\otimes}_{\mathcal{C}}$, but this is not essential.) The additive quotient $\mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{I}$ amounts to quotienting out all objects $[2]^{\oplus m}$ for $m \geq 1$. The resulting category $\mathcal{C}/\mathcal{I}$ consists of two copies of $\Bbbk$-vector spaces, one generated by the image of $[1]$ and one by the image of $[3] = \Sigma[1]$, and its tensor product is forced by $[3] \otimes [3] \simeq [1]$. In other words, $\mathcal{C}/\mathcal{I}$ is the category of finite dimensional $\mathbb{Z}/2$-graded $\Bbbk$-vector spaces. We saw in Proposition 1.5 that a separable commutative ring-object in that category must be concentrated in degree zero, *i.e.* the image of A in $\mathcal{C}/\mathcal{I}$ contains no copy of $[3]$. □

4.4. **Proposition.** *Let $N \lhd G$ be a normal subgroup of a finite group G and assume that p divides the order of N. Consider $\pi : G \twoheadrightarrow \bar{G} = G/N$ the corresponding quotient. Then inflation $\mathrm{Infl}^{\bar{G}}_{G} = \pi^* : \Bbbk\bar{G}\text{-}\mathrm{mod} \to \Bbbk G\text{-}\mathrm{stmod}$, from the* ordinary *module category of $\bar{G}$ to the* stable *category of G, is fully faithful and its essential image consists of those objects isomorphic in $\Bbbk G\text{-}\mathrm{stmod}$ to some $\Bbbk G$-module M such that $\mathrm{Res}^G_N M$ has trivial N-action.*

4.5. *Remark.* Objects of $\Bbbk G\text{-}\mathrm{stmod}$ are the same as those of $\Bbbk G\text{-}\mathrm{mod}$, *i.e.* finitely generated $\Bbbk G$-modules. However, the property "$\mathrm{Res}^G_N M$ has trivial N-action" is not stable under isomorphism in $\Bbbk G\text{-}\mathrm{stmod}$, since one can add to M a projective $\Bbbk G$-module. This explains the phrasing of the above statement.

Proof of Proposition 4.4. The image of $\mathrm{Infl}^{\bar{G}}_{G} : \Bbbk\bar{G}\text{-}\mathrm{mod} \to \Bbbk G\text{-}\mathrm{mod}$ consists precisely of those $\Bbbk G$-modules on which N acts trivially. This gives the statement about the essential image by taking closure under isomorphism in $\Bbbk G\text{-}\mathrm{stmod}$. To show that $\mathrm{Infl}^{\bar{G}}_{G} : \Bbbk\bar{G}\text{-}\mathrm{mod} \to \Bbbk G\text{-}\mathrm{stmod}$ is fully faithful, note first that it is full because both $\mathrm{Infl}^{\bar{G}}_{G} : \Bbbk\bar{G}\text{-}\mathrm{mod} \to \Bbbk G\text{-}\mathrm{mod}$ and $\Bbbk G\text{-}\mathrm{mod} \twoheadrightarrow \Bbbk G\text{-}\mathrm{stmod}$ are full. For faithfulness, consider the commutative diagram

$$\begin{tikzcd}
\Bbbk G\text{-}\mathrm{stmod} \arrow[d, "\mathrm{Res}^G_N"'] & \Bbbk G\text{-}\mathrm{mod} \arrow[l, two heads] & \Bbbk\bar{G}\text{-}\mathrm{mod} \arrow[l, "\mathrm{Infl}^{\bar{G}}_{G}"'] \arrow[d, "\text{faithf.}"] \\
\Bbbk N\text{-}\mathrm{stmod} & & \Bbbk\text{-}\mathrm{mod} \arrow[ll, "\text{faithf.}"']
\end{tikzcd}$$

As the right-hand and bottom functors are faithful, so is the top composite. □

Proof of Theorem 4.1. Let A be an indecomposable tt-ring in $\Bbbk G\text{-}\mathrm{stmod}$, where $G = C_{p^n}$. We proceed by induction on n. The case $n = 1$ was settled in Theorem 3.1 so we assume $n \geq 2$. We can choose a $\Bbbk G$-module M representing the object A in $\Bbbk G\text{-}\mathrm{stmod}$ and therefore assume that M has no projective summand.

Consider $N = C_p \lhd G$ the unique cyclic subgroup of order p. We claim that $\mathrm{Res}^G_N M$ has trivial C_p-action (in $\Bbbk C_p\text{-}\mathrm{mod}$).

Suppose first that $p = 2$. We need to prove that $\operatorname{Res}^G_{C_2} M$ does not contain a projective summand. By Proposition 4.3, we know that $\operatorname{Res}^G_{C_4} M$ has no indecomposable summand [3] of dimension 3. By Proposition 4.2 for $m = 2$, we know therefore that $\operatorname{Res}^G_{C_4} M$ has no projective factor either. Hence, $\operatorname{Res}^G_{C_4} M$ consists of a sum of copies of [1] and [2], which both restrict to trivial modules over C_2.

Suppose now that p is odd. Then we can use Proposition 4.2 directly from G to C_p (*i.e.* take $m = 1$). We know that $\operatorname{Res}^G_{C_p} M$ consists only of trivial $\Bbbk C_p$-modules and possibly projectives, since its class in the stable category is trivial by Theorem 3.1. None of those indecomposable summands have dimension $p - 1$. Hence, Proposition 4.2 tells us that $\operatorname{Res}^G_{C_p} M$ has no projective summand.

So, we have proved the claim: $\operatorname{Res}^G_{C_p} M$ has no projective summand. On the other hand, by Theorem 3.1, this $\operatorname{Res}^G_{C_p} M$ is trivial in the stable category. Combining both facts, we have shown that $\operatorname{Res}^G_{C_p} M$ has trivial $\Bbbk C_p$-action. By Proposition 4.4 for our $N = C_p \lhd G$, it follows that the image A of M in $\Bbbk G\text{-}\operatorname{stmod}$ belongs to the essential image of the fully faithful tensor functor $\Bbbk\bar{G}\text{-}\operatorname{mod} \to \Bbbk G\text{-}\operatorname{stmod}$ where $\bar{G} = G/N$. By the much easier ordinary module category case (see [Bal15, Rem. 4.6]), we see that $A \simeq \pi^*\Bbbk X \simeq \Bbbk\pi^* X$ for some finite $\bar{G}$-set X, which can then be viewed as a finite G-set $\pi^* X$ through $\pi : G \to \bar{G}$. $\square$

References

[AF78] Gert Almkvist and Robert Fossum. Decomposition of exterior and symmetric powers of indecomposable $\mathbf{Z}/p\mathbf{Z}$-modules in characteristic p and relations to invariants. In *Séminaire d'Algèbre Paul Dubreil, 30ème année (Paris, 1976–1977)*, volume 641 of *Lecture Notes in Math.*, pages 1–111. Springer, Berlin, 1978.

[AG60] Maurice Auslander and Oscar Goldman. The Brauer group of a commutative ring. *Trans. Amer. Math. Soc.*, 97:367–409, 1960.

[Bal11] Paul Balmer. Separability and triangulated categories. *Adv. Math.*, 226(5):4352–4372, 2011.

[Bal15] Paul Balmer. Stacks of group representations. *J. Eur. Math. Soc. (JEMS)*, 17(1):189–228, 2015.

[Bal16] Paul Balmer. The derived category of an étale extension and the separable Neeman-Thomason theorem. *J. Inst. Math. Jussieu*, 15(3):613–623, 2016.

[BBW09] G. Böhm, T. Brzeziński, and R. Wisbauer. Monads and comonads on module categories. *J. Algebra*, 322(5):1719–1747, 2009.

[BC86] D. J. Benson and J. F. Carlson. Nilpotent elements in the Green ring. *J. Algebra*, 104(2):329–350, 1986.

[BDS15] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. Restriction to finite-index subgroups as étale extensions in topology, KK-theory and geometry. *Algebr. Geom. Topol.*, 15(5):3025–3047, 2015.

[BV07] A. Bruguières and A. Virelizier. Hopf monads. *Adv. Math.*, 215(2):679–733, 2007.

[CFP08] Jon F. Carlson, Eric M. Friedlander, and Julia Pevtsova. Modules of constant Jordan type. *J. Reine Angew. Math.*, 614:191–234, 2008.

[CJ64] Peter Crawley and Bjarni Jónsson. Refinements for infinite direct decompositions of algebraic systems. *Pacific J. Math.*, 14:797–855, 1964.

[CR66] C. W. Curtis and I. Reiner, *Representations of finite groups and associated algebras* Interscience Publishers, New York, 1966.

[DI71] F. DeMeyer and E. Ingraham. *Separable algebras over commutative rings.* Lecture Notes in Mathematics, Vol. 181. Springer-Verlag, Berlin, 1971.

[Kel64] G. M. Kelly. On the radical of a category. *J. Austral. Math. Soc.*, 4:299–307, 1964.

[Lam91] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991.

[Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Adv. Math.*, 291:403–541, 2016.

[Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, 2001.

[Nee15] Amnon Neeman. Separable monoids in $\mathrm{D}_{\mathrm{qc}}(X)$. *J. Reine Angew. Math.*, 2015. Preprint, 44 pages, to appear, available at http://dx.doi.org/10.1515/crelle-2015-0039.

[Pre91] Alexander Premet. The green ring of a simple three-dimensional lie p-algebra (russian). *Izv. Vyssh. Uchebn. Zaved. Mat.*, 10:56–67, 1991.

[Ric97] Jeremy Rickard. Idempotent modules in the stable category. *J. London Math. Soc. (2)*, 56(1):149–170, 1997.

[War69] R. B. Warfield, Jr. A Krull-Schmidt theorem for infinite sums of modules. *Proc. Amer. Math. Soc.*, 22:460–465, 1969.

Paul Balmer, Mathematics Department, UCLA, Los Angeles, CA 90095-1555, USA
E-mail address: balmer@math.ucla.edu
URL: http://www.math.ucla.edu/∼balmer

Jon Carlson, Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: jfc@math.uga.edu
URL: http://www.math.uga.edu/~jfc