SEPARABLE EXTENSIONS IN TENSOR-TRIANGULAR GEOMETRY AND GENERALIZED QUILLEN STRATIFICATION

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ABSTRACT. We exhibit a link between the Going-Up Theorem in commutative algebra and Quillen Stratification in modular representation theory. To this effect, we study the continuous map induced on spectra by a separable extension of tensor-triangulated categories. We determine the image of this map and bound the cardinality of its fibers by the degree of the extension. We then prove a weak form of descent, "up-to-nilpotence", which allows us to generalize Quillen Stratification to equivariant derived categories.

Nous exhibons un lien entre le théorème du "going-up" en algèbre commutative et le théorème de stratification de Quillen en théorie des représentations modulaires. Dans ce but, nous étudions l'application continue induite sur les spectres par une extension séparable de catégories triangulées tensorielles. Nous en déterminons l'image et bornons le cardinal de ses fibres par le degré de l'extension. Nous prouvons alors une forme faible de descente, "à nilpotence près", qui nous permet de généraliser la stratification de Quillen à d'autres catégories dérivées équivariantes.

CONTENTS

INTRODUCTION

There are rich and growing connections between commutative algebra and modular representation theory, notably via homological methods. Some of these connections can be built using tensor-triangulated categories. Recall that tensor-triangulated categories also appear in several other settings, like motivic theory, equivariant stable homotopy theory, or Kasparov's KK-theory of C*-algebras, for instance. This framework is the backdrop of *tensor-triangular geometry*, or *tt-geometry* for short, see [Bal10].

In the present work, we use tt-geometry to connect two classical and well-known results, namely the Going-Up Theorem in commutative algebra and Quillen's Stratification Theorem in modular representation theory. Let us first remind the reader:

Theorem (Going-Up). *Let $R \subset A$ be an integral extension of commutative rings, let $\mathfrak{q}$ be a prime in A and $\mathfrak{p}'$ a prime in R containing $\mathfrak{q} \cap R$; then there exists a prime $\mathfrak{q}'$ in A containing $\mathfrak{q}$ such that $\mathfrak{q}' \cap R = \mathfrak{p}'$. Further, $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is surjective and a weak form of injectivity holds, known as "Incomparability": if $\mathfrak{q} \subseteq \mathfrak{q}'$ are two primes in A such that $\mathfrak{q} \cap R = \mathfrak{q}' \cap R$ then $\mathfrak{q} = \mathfrak{q}'$.*

Theorem (Quillen Stratification [Qui71]). *Let G be a finite group and $\Bbbk$ be a field of positive characteristic p dividing the order of G. Let $\mathcal{V}_G := \mathrm{Proj}(\mathrm{H}^\bullet(G, \Bbbk))$ be the* projective support variety *of G. Then there is a canonical homeomorphism*

$$(0.1) \qquad \underset{H \in \mathrm{Or}(G,\,\mathrm{Abelem})}{\mathrm{colim}} \mathcal{V}_H \ \xrightarrow{\sim} \ \mathcal{V}_G$$

where $\mathrm{Or}(G, \mathrm{Abelem})$ is the full subcategory of the orbit category of G on the elementary abelian p-subgroups of G (see Definition 4.6 if necessary).

It is not obvious to the naked eye why the above two results should be related. Let us observe the tip of the iceberg: The Going-Up Theorem forces the rings A and R to have the same Krull dimension; similarly Quillen Stratification forces the Krull dimension of $\mathcal{V}_G$ to be the maximum of the dimensions of the $\mathcal{V}_H$, i.e. the p-rank of G minus one – an important application of [Qui71].

Here, we prove an analogue of Going-Up in tt-geometry and show that it specializes to Quillen Stratification when applied to modular representation theory. This result illustrates the connections between the two subjects and the depth of tt-geometry. To go beyond unification and connection, we also prove some new results, namely we extend Quillen Stratification to any tensor-triangulated category receiving the derived category of G.

1. STATEMENT OF RESULTS

To do tt-geometry, one needs a tt-category $\mathcal{K}$. (Here, "tt" is short for "tensor-triangular" or "tensor-triangulated", as appropriate.) In commutative algebra, we use $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(R)$, the homotopy category of bounded complexes of finitely generated projective R-modules. In modular representation theory, we use $\mathcal{K} = \mathrm{stab}(\Bbbk G)$, the stable category of finitely generated $\Bbbk G$-modules modulo the projective ones.

To recover spaces like the affine scheme $\mathrm{Spec}(R)$ in the first example and the support variety $\mathcal{V}_G$ in the second, we use the *spectrum* $\mathrm{Spc}(\mathcal{K})$ of a tt-category $\mathcal{K}$. This fundamental tool of tt-geometry was introduced in [Bal05] as the universal topological space in which objects x of $\mathcal{K}$ admit reasonable supports $\mathrm{supp}(x) \subseteq \mathrm{Spc}(\mathcal{K})$; see Rem. 2.6. The spectrum can also be constructed by means of *prime ideals* $\mathcal{P}$ in $\mathcal{K}$. By [Bal05], it recovers the Zariski spectrum $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R)) \cong \mathrm{Spec}(R)$ and the support variety $\mathrm{Spc}(\mathrm{stab}(\Bbbk G)) \cong \mathcal{V}_G$. This unification is the key to relate affine schemes and support varieties via tt-geometry. See more in Remark 2.7.

To exhibit a first analogy between Going-Up and Quillen Stratification, let us observe that both situations involve not only *one* tt-category but actually *two* (or more). In commutative algebra, it is rather obvious: We have the ring homomorphism $R \to A$, hence an extension-of-scalars $\mathrm{D}^{\mathrm{perf}}(R) \to \mathrm{D}^{\mathrm{perf}}(A)$ which is

a tt-functor. In modular representation theory, we also have tt-functors $\mathrm{Res}_H^G :$ $\mathrm{stab}(\Bbbk G) \to \mathrm{stab}(\Bbbk H)$ given by restriction from the group G to its subgroups $H \leq G$, for instance the elementary abelian ones. Another key step in our discussion is to understand those restriction functors $\mathrm{stab}(\Bbbk G) \to \mathrm{stab}(\Bbbk H)$ as a form of "extension-of-scalars", similar to the extension-of-scalars of commutative algebra.

This is made possible thanks to a good notion of "ring" in a general tt-category $\mathcal{K}$, inspired by commutative algebra but flexible enough to be useful in representation theory as well. These good rings are the "tt-rings" of [Bal14] : An associative and unital ring object $A \otimes A \xrightarrow{\mu} A$ in $\mathcal{K}$ is called a *tt-ring* if it is commutative and *separable*, in the classical sense (i.e. the multiplication μ has a section $\sigma : A \to A \otimes A$ which is A-linear on both sides). Separability of A guarantees that the category $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ of good-old A-modules in $\mathcal{K}$ remains triangulated, by [Bal11] (see Remark 2.14). Commutativity of A allows us to equip $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ with a tensor $\otimes_A$. In short, if A is a tt-ring in a tt-category $\mathcal{K}$, then $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ is again a tt-category and extension-of-scalars $F_A : \mathcal{K} \longrightarrow A\text{-}\mathrm{Mod}_{\mathcal{K}}$ is a tt-functor.

By [Bal15] the restriction functor $\mathrm{Res}_H^G : \mathrm{stab}(\Bbbk G) \to \mathrm{stab}(\Bbbk H)$ is actually such an extension-of-scalars with respect to the particular tt-ring $A_H^G := \Bbbk(G/H)$, with multiplication extending $\Bbbk$-linearly the rule $\gamma \cdot \gamma = \gamma$ and $\gamma \cdot \gamma' = 0$ for all $\gamma \neq \gamma'$ in G/H (see Constr. 4.1). More precisely, there is an equivalence

$$(1.1) \qquad\qquad A_H^G\text{-}\mathrm{Mod}_{\mathcal{K}(G)} \cong \mathcal{K}(H)$$

for $\mathcal{K}(G) = \mathrm{stab}(\Bbbk G)$ and of course $\mathcal{K}(H) = \mathrm{stab}(\Bbbk H)$; under this equivalence, extension-of-scalars $F_{A_H^G}$ becomes isomorphic to restriction Res_H^G. In particular, the induced map on spectra recovers the usual map on support varieties $\mathcal{V}_H \to \mathcal{V}_G$. This recasting of restriction as an extension-of-scalars is already true for derived categories, i.e. the above (1.1) holds if $\mathcal{K}(G)$ means $\mathrm{D}^b(\Bbbk G)$, etc. See details in [Bal15, Part I]. Furthermore we show in [BDS14] that such results hold quite generally for equivariant tt-categories, way beyond representation theory.

In general, for any tt-category $\mathcal{K}$ and any tt-ring A in $\mathcal{K}$, the tt-functor $F_A : \mathcal{K} \to A\text{-}\mathrm{Mod}_{\mathcal{K}}$ induces a continuous map on spectra

$$(1.2) \qquad\qquad \varphi_A := \mathrm{Spc}(F_A) \ : \quad \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \longrightarrow \mathrm{Spc}(\mathcal{K}) \,.$$

It is clearly important to study this map φ_A in general, independently of Going-Up or Quillen Stratification. Here is our main tt-geometric result (see details below):

1.3. **Theorem** (Descent-up-to-nilpotence, see Thm. 3.19)**.** *Let $\mathcal{K}$ be a tt-category and let A be a tt-ring in $\mathcal{K}$. Suppose that A has finite degree and that A detects $\otimes$-nilpotence of morphisms. Then we have a coequalizer of topological spaces*

$$(1.4) \qquad \mathrm{Spc}(A^{\otimes 2}\text{-}\mathrm{Mod}_{\mathcal{K}}) \ \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} \ \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \xrightarrow{\varphi_A} \mathrm{Spc}(\mathcal{K})$$

where φ_1 and φ_2 are induced by the two obvious homomorphisms $A \rightrightarrows A \otimes A$.

The *degree* of a tt-ring A has also been introduced in [Bal14] but can easily be considered as a black-box here. It is the natural measure of the "size" of the tt-ring A. It will allow us to prove some results by induction on the degree, via Theorem 2.19. Having finite degree is a mild hypothesis which holds for all tt-rings in all standard tt-categories of compact objects in algebraic geometry, homotopy theory or modular representation theory, by [Bal14, § 4]. See more in Section 2.

The assumption that A *detects $\otimes$-nilpotence* is the important condition of the theorem. It means that if a morphism f in $\mathcal{K}$ satisfies $A \otimes f = 0$ then $f^{\otimes n} = 0$ for some $n \geq 1$. This property of the tt-ring A will be called *nil-faithfulness*. Faithfulness ($A \otimes f = 0 \Rightarrow f = 0$) is a special case of interest, as we shall discuss below. Proposition 3.15 provides further equivalent characterizations of nil-faithfulness, like for instance that A generates $\mathcal{K}$ as a thick $\otimes$-ideal, i.e. $\mathrm{supp}(A) = \mathrm{Spc}(\mathcal{K})$.

The strength of (1.4) will be discussed below, when we return to examples. For the moment, simply note that it tells us that φ_A is onto and that two points of $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ have the same image only if they are the image of the same point under φ_1 and φ_2. This result is as good as it gets in full tt-generality. In fact, Theorem 1.3 is the culmination of a series of results of independent interest, proved in Section 3 and summarized in Theorem 1.5 below. They illustrate how our study of φ_A in tt-geometry mirrors integral extensions in commutative algebra.

1.5. Theorem. *Let A be a tt-ring in a tt-category $\mathcal{K}$ and consider φ_A as in (1.2).*

(a) *The image of φ_A equals the support of A, that is, $\mathrm{Im}(\varphi_A) = \mathrm{supp}(A)$ in $\mathrm{Spc}(\mathcal{K})$. It will follow that φ_A is onto if and only if A is nil-faithful.*

Suppose furthermore that A has finite degree d.

(b) *Going-Up : For every $\mathcal{Q} \in \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ and every $\mathcal{P}' \in \overline{\{\varphi_A(\mathcal{Q})\}}$ there exists $\mathcal{Q}' \in \overline{\{\mathcal{Q}\}}$ such that $\varphi_A(\mathcal{Q}') = \mathcal{P}'$.*

(c) *Incomparability : If $\mathcal{Q}' \in \overline{\{\mathcal{Q}\}}$ in $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ are such that $\varphi_A(\mathcal{Q}) = \varphi_A(\mathcal{Q}')$ then $\mathcal{Q} = \mathcal{Q}'$.*

(d) *The fibers of φ_A are finite and discrete, with at most d points.*

The result implies that $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ and $\mathrm{supp}(A)$ have the same Krull dimension. If A is nil-faithful then $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ and $\mathrm{Spc}(\mathcal{K})$ have the same Krull dimension.

As we shall see, the proofs of these results are quite different from their commutative algebra counterparts. The main obstacle to the tt-generalization of classical techniques is the absence of a tt-quotient, i.e. the tt-equivalent of the ring R/I for an ideal $I \subset R$. (Verdier quotients $\mathcal{K}/\mathcal{J}$ lead to open subschemes, not to closed ones.) In particular, there is no "residue field" in general tt-geometry, at least for the moment, and this makes the description of the fibers of φ_A particularly difficult.

To conclude our discussion, we still want to explain why Theorem 1.3 is called "descent-up-to-nilpotence" and understand nil-faithfulness in examples.

Conceptually, the category $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ should be approached as a *simplification* of the original category $\mathcal{K}$. In some sense, it represents a "finite étale extension of $\mathcal{K}$". This idea goes hand-in-hand with Grothendieck's theory of *descent*, i.e. the problem of reconstructing the original category $\mathcal{K}$ from $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ and data over $A^{\otimes 2}$ and $A^{\otimes 3}$. By [Bal12], if A was not just nil-faithful but really faithful ($A \otimes f = 0 \Rightarrow f = 0$) then A would satisfy descent in $\mathcal{K}$, which roughly says that

$$\mathcal{K} \longrightarrow A\text{-}\mathrm{Mod}_{\mathcal{K}} \rightrightarrows A^{\otimes 2}\text{-}\mathrm{Mod}_{\mathcal{K}}$$

is an "equalizer of categories" (plus cocycle conditions over $A^{\otimes 3}$). In that case, it is easier to prove, and even easier to believe, that the contravariant $\mathrm{Spc}(-)$ turns the latter "equalizer" into the coequalizer of spaces (1.4). The remarkable fact about Theorem 1.3 is that (1.4) is a coequalizer even when A is not faithful but only *nil*-faithful, despite failure of descent in this situation! This explains the idea of "descent-up-to-nilpotence".

To fully appreciate the difference between faithful and nil-faithful, i.e. between descent and descent-up-to-nilpotence, we turn to our example of modular representation theory, where we will also better understand the coequalizer (1.4).

Let us test the distinction between faithful and nil-faithful on the tt-rings $A_H^G = \Bbbk(G/H)$ that we discussed above, see (1.1). We assume for simplicity that G is a p-group, where $p = \mathrm{char}(\Bbbk) > 0$. Then consider in $\mathcal{K} = \mathrm{stab}(\Bbbk G)$, the tt-ring

$$A := \Bbbk(G/H_1) \times \cdots \times \Bbbk(G/H_n) \qquad \text{for subgroups } H_1, \ldots, H_n \leq G \,.$$

This A is faithful only if one of the H_i equals G, which is sadly restrictive since $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ contains a copy of $\mathcal{K}$ in that case. On the other hand, this same A is nil-faithful much more often, most famously if $H_1, \ldots, H_n$ are the elementary abelian p-subgroups of G. In fact, nil-faithfulness of this particular tt-ring A is Serre's old theorem on the vanishing of the Bocksteins [Ser65]; see Thm. 4.3. In other words, descent happens only in the trivial case where $\mathcal{K}$ is a summand of $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ but descent-up-to-nilpotence happens much more often. In fact, in that case, our descent-up-to-nilpotence Theorem 1.3 recovers Quillen Stratification.

Although our goal is *not* to give a new proof of Quillen's result, we hope that recovering such a major result as an example could help the reader grasp the power of Theorem 1.3. To provide new applications, we extend Quillen Stratification to every tt-category receiving the derived category. This is Theorem 4.10 below:

1.6. **Theorem.** *Let p be a prime and G a finite group with p dividing $|G|$. Let $\mathcal{K}(G)$ be a tt-category which admits a tt-functor $\Phi : \mathrm{D}^{\mathrm{b}}(\mathbb{F}_p G) \longrightarrow \mathcal{K}(G)$, where $\mathbb{F}_p$ is the finite field with p elements. For every subgroup $H \leq G$, consider the tt-ring $A_H = \Phi(\mathbb{F}_p(G/H))$, image in $\mathcal{K}(G)$ of the complex $\mathbb{F}_p(G/H) \in \mathrm{D}^{\mathrm{b}}(\mathbb{F}_p G)$ concentrated in degree zero. Define $\mathcal{K}(H) = A_H\text{-}\mathrm{Mod}_{\mathcal{K}(G)}$ the corresponding category of modules. Then the tt-functors $F_{A_H} : \mathcal{K}(G) \longrightarrow \mathcal{K}(H)$ induce a homeomorphism*

$$(1.7) \qquad\qquad \operatorname*{colim}_{H \in \mathrm{Or}(G,\,\mathrm{Abelem})} \mathrm{Spc}(\mathcal{K}(H)) \; \xrightarrow{\;\sim\;} \; \mathrm{Spc}(\mathcal{K}(G)) \,.$$

The reason for the notation $\mathcal{K}(H)$ is that, in most examples, the category $\mathcal{K}(G)$ is defined for all finite groups G together with an equivalence $A_H\text{-}\mathrm{Mod}_{\mathcal{K}(G)} \cong \mathcal{K}(H)$, as in (1.1). As we show in [BDS14], such "monadicity" results happen in most equivariant settings, way beyond representation theory. They give (1.7) its full flavor since in that case the left-hand category $\mathcal{K}(H)$ is not merely a category of modules in $\mathcal{K}(G)$ but the true object of study over the subgroup H. To give an example for which $\mathrm{Spc}(\mathcal{K}(G))$ is not known, one could take $\mathcal{K}(G)$ to be the bounded derived category of equivariant vector bundles over a variety with G-action. In that case, Quillen Stratification is new. See Example (4.14).

The organization of the paper should now be clear from the table of content.

2. Recalling tt-rings and their degree

2.1. *Convention.* We use the word *triangulated* in the sense of [Bal11, §5], or Künzer [Kün07] and Maltsiniotis [Mal06]. This is a mild strengthening of Verdier's traditional definition in which we require octahedra to satisfy the analogue of the morphism axiom. One can also require higher triangles but we do not need them here. All examples (stable homotopy categories) are triangulated in that sense.

2.2. *Definition.* A *tt-category* $\mathcal{K}$ is an essentially small, idempotent-complete tensor-triangulated category. Hence $\mathcal{K}$ is triangulated, all idempotent endomorphisms in $\mathcal{K}$ split, and $\mathcal{K}$ is equipped with a symmetric monoidal structure $\otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}$ such that $x \otimes - : \mathcal{K} \to \mathcal{K}$ is exact for all objects $x \in \mathcal{K}$. We denote by $\mathbb{1} \in \mathcal{K}$ its $\otimes$-unit. Throughout this paper, $\mathcal{K}$, $\mathcal{L}$, ... denote tt-categories. A *tt-functor* $F : \mathcal{K} \to \mathcal{L}$ is a functor which is both exact and monoidal, in such a way that the two natural isomorphisms from $F(x \otimes \Sigma y)$ to $\Sigma F(x \otimes y)$ agree (using in different order the compatibility of F with $\otimes$, of F with Σ, and of $\otimes$ with Σ) and similarly on the left.

2.3. *Example.* Let X be a quasi-compact and quasi-separated scheme (e. g. a noetherian scheme or an affine scheme) then the derived category $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ of perfect complexes [BGI71] is a tt-category. For $X = \mathrm{Spec}(R)$, this is the tt-category $\mathrm{D}^{\mathrm{perf}}(R)$ of the introduction.

2.4. *Example.* Let G be a finite group and $\Bbbk$ a field of positive characteristic p dividing $|G|$. Then the derived category $\mathrm{D}^{\mathrm{b}}(\Bbbk G) := \mathrm{D}^{\mathrm{b}}(\Bbbk G - \mathrm{mod})$ and the stable category $\mathrm{stab}(\Bbbk G)$ of finitely generated $\Bbbk G$-modules are tt-categories and the well-known (Buchweitz-Rickard) quotient $\mathrm{D}^{\mathrm{b}}(\Bbbk G) \twoheadrightarrow \mathrm{stab}(\Bbbk G)$ is a tt-functor. See [Ric89].

2.5. *Notation.* We write $x \leq y$ when x is a direct summand of y. A triangulated subcategory $\mathcal{J} \subseteq \mathcal{K}$ is called a *thick $\otimes$-ideal* if $x \leq y \in \mathcal{J} \Rightarrow x \in \mathcal{J}$ and if $\mathcal{K} \otimes \mathcal{J} \subseteq \mathcal{J}$. We denote by $\langle \mathcal{E} \rangle$ the thick $\otimes$-ideal generated by a collection of objects $\mathcal{E} \subseteq \mathcal{K}$.

2.6. *Remark.* We do not need much tt-geometry here but we simply recall that the *spectrum* $\mathrm{Spc}(\mathcal{K})$ of a tt-category $\mathcal{K}$ consists of all *prime* thick $\otimes$-ideals $\mathcal{P} \subsetneq \mathcal{K}$, i.e. such that $x \otimes y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The topology of $\mathrm{Spc}(\mathcal{K})$ has an open basis composed of all subsets $\mathcal{U}(x) := \big\{ \mathcal{P} \,\big|\, x \in \mathcal{P} \big\}$, for $x \in \mathcal{K}$. The closed complement $\mathrm{supp}(x) := \big\{ \mathcal{P} \,\big|\, x \notin \mathcal{P} \big\}$ is the *support* of x in $\mathrm{Spc}(\mathcal{K})$.

2.7. *Remark.* Determining tt-spectra is an ongoing enterprise in several fields of mathematics, with direct link to the classifications of thick $\otimes$-ideals. This program originated in stable homotopy theory [HS98], and extended to algebraic geometry [Hop87, Nee92, Tho97] and modular representation theory [BCR97, FP07] prior to [Bal05]. More recent progress has been made in noncommutative topology [Del10], for Artin-Tate motives [Pet13], for noncommutative motives [DT12], for group rings over commutative algebras [Ste13], for category algebras [Xu14] and for Lie superalgebras [BKN14]. Still, computing $\mathrm{Spc}(\mathcal{K})$ remains an open challenge for many important tt-categories $\mathcal{K}$, like equivariant derived categories or equivariant stable homotopy categories [GM95].

One basic tt-result that we shall need a few times is the following :

2.8. **Lemma** (Existence Lemma [Bal05, Lem. 2.2]). *Let $\mathcal{K}$ be a tt-category, $\mathcal{J} \subseteq \mathcal{K}$ a thick $\otimes$-ideal and $S \subset \mathcal{K}$ a $\otimes$-multiplicative collection of objects ($\mathbb{1} \in S \supseteq S \otimes S$) such that $S \cap \mathcal{J} = \varnothing$. Then there exists $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ such that $\mathcal{J} \subseteq \mathcal{P}$ and $\mathcal{P} \cap S = \varnothing$.*

2.9. *Notation.* Given n objects $x_1, \ldots, x_n$ in $\mathcal{K}$ and a permutation $\pi \in S_n$, we also denote by $\pi : x_1 \otimes \cdots \otimes x_n \overset{\sim}{\to} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)}$ the induced isomorphism.

2.10. *Definition.* A *ring object* A in $\mathcal{K}$ is a monoid $(A, \mu : A \otimes A \to A, \eta : \mathbb{1} \to A)$, i.e. admits associative multiplication $\mu : A \otimes A \to A$ and two-sided unit η. It is *commutative* if $\mu \circ (12) = \mu$. A *morphism of (commutative) ring objects*, or simply *a homomorphism*, is a morphism $f : A \to B$ in $\mathcal{K}$, compatible with multiplications and units. We also say that B is an *A-algebra* or that B is a *ring over A*.

2.11. *Notation.* For two ring objects A and B, the ring object $A \times B$ is $A \oplus B$ with component-wise structure. On the other hand, the ring object $A \otimes B$ has multiplication $(\mu_1 \otimes \mu_2) \circ (23) : (A \otimes B)^{\otimes 2} \longrightarrow A \otimes B$ and obvious unit.

2.12. *Definition.* A ring object A is called *separable* if there exists a morphism $\sigma : A \to A \otimes A$ such that $\mu\sigma = \mathrm{id}_A$ and $(1 \otimes \mu) \circ (\sigma \otimes 1) = \sigma\mu = (\mu \otimes 1) \circ (1 \otimes \sigma) : A^{\otimes 2} \to A^{\otimes 2}$. This simply means that A is projective as A, A-bimodule.

2.13. *Remark.* If A and B are separable then so are $A \times B$ and $A \otimes B$.

2.14. *Remark.* A *(left) A-module* is a pair (x, ρ) where $x \in \mathcal{K}$ and where the so-called action $\rho : A \otimes x \to x$ satisfies the usual associativity and unit conditions. We denote by $A\text{-}\mathrm{Mod}_\mathcal{K}$ the category of A-modules with A-linear morphisms and by $F_A : \mathcal{K} \to A\text{-}\mathrm{Mod}_\mathcal{K}$ the functor $F_A(y) = (A \otimes y, \mu \otimes 1)$, etc. For details see [EM65] or [Bal11].

By [Bal11], when A is separable, the category of A-modules in $\mathcal{K}$ remains triangulated in such a way that both the extension-of-scalars functor $F_A : \mathcal{K} \longrightarrow A\text{-}\mathrm{Mod}_\mathcal{K}$ and the forgetful functor $U_A : A\text{-}\mathrm{Mod}_\mathcal{K} \longrightarrow \mathcal{K}$ are exact. We really mean "modules in the homotopy category" here, not "homotopy category of modules", thanks to separability! The key fact is that the co-unit $\epsilon : F_A U_A \to \mathrm{Id}_{A\text{-}\mathrm{Mod}_\mathcal{K}}$ is split surjective. In particular, every A-module x is a direct summand of a free one: $x \leq F_A U_A(x)$.

The following simple definition [Bal14] plays a central role in our theory.

2.15. *Definition.* A ring object A is a *tt-ring* if A is commutative and separable.

The terminology is chosen so that if A is a tt-ring in a tt-category $\mathcal{K}$ then $A\text{-}\mathrm{Mod}_\mathcal{K}$ remains a tt-category. See details in [Bal14], notably about the tensor structure $\otimes_A$ *over A* on the category $A\text{-}\mathrm{Mod}_\mathcal{K}$. The projection formula [Bal14, Prop. 1.2] says that $U_A(x \otimes_A F_A(y)) \cong U_A(x) \otimes y$ for all $x \in A\text{-}\mathrm{Mod}_\mathcal{K}$ and $y \in \mathcal{K}$.

2.16. *Remark.* In commutative algebra, an R-algebra A will not be a tt-ring in $\mathrm{D}^{\mathrm{perf}}(R)$ in general, unless A is finite étale over R. In that case, the tt-category $\mathrm{D}^{\mathrm{perf}}(A)$ coincides with $A\text{-}\mathrm{Mod}_{\mathrm{D}^{\mathrm{perf}}(R)}$ by [Bal11, Cor. 6.6]. Thus the reader might prefer to think of our extensions-of-scalars $F_A : \mathcal{K} \to A\text{-}\mathrm{Mod}_\mathcal{K}$ along tt-rings as finite étale morphisms.

2.17. *Remark.* Suppose that $\mathcal{K} = \mathcal{T}^c$ consists of the compact objects of a compactly generated (big) tensor-triangulated category $\mathcal{T}$. Then any smashing localization of $\mathcal{T}$ is in fact an extension-of-scalars with respect to a tt-ring A in $\mathcal{T}$, which is moreover an idempotent. Conversely, any ring object such that $\mu : A \otimes A \to A$ is an isomorphism yields a smashing localization. This provides another class of examples of separable extensions. In this paper, we restrict attention to compact tt-rings $A \in \mathcal{K}$, in order to speak of $\mathrm{supp}(A)$, thus excluding smashing localizations. Yet, some of our results might extend to "big" tt-rings $A \in \mathcal{T}$ with a suitable notion of "big support".

2.18. *Remark.* We refer to [Bal14] for the notion of the *degree* of a tt-ring A. We use it as a black box here, emphasizing that we do not know any tt-ring of infinite degree. (Even the big smashing rings of Remark 2.17 actually have degree 1.) In particular the "equivariant" tt-rings $\Bbbk(G/H)$ of [Bal15] are of finite degree $[G:H]$ in $\mathrm{D}^{\mathrm{b}}(\Bbbk G)$. Furthermore, the degree cannot increase along tt-functors. The key result allowing induction on the degree is the following [Bal14, Thm. 3.6]:

2.19. Theorem. *Let A be a tt-ring of finite degree d in a tt-category $\mathcal{K}$. Then there exists a tt-ring C and a homomorphism $g : A \otimes A \to C$ such that $\left(\begin{smallmatrix} \mu \\ g \end{smallmatrix}\right) : A \otimes A \xrightarrow{\sim} A \times C$ is an isomorphism. Moreover, C is of degree $d - 1$ as a tt-ring in $A\text{-}\mathrm{Mod}_{\mathcal{K}}$, when viewed as an A-algebra through $g \circ (1 \otimes \eta) : A \to C$.*

3. Descent up-to-nilpotence

3.1. Notation. Given a tt-ring A in a tt-category $\mathcal{K}$, we have a continuous (and spectral) map $\varphi_A := \mathrm{Spc}(F_A) : \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \longrightarrow \mathrm{Spc}(\mathcal{K})$, as in (1.2). It is defined by $\varphi_A(\mathcal{P}) = F_A^{-1}(\mathcal{P})$ for every prime $\mathcal{P} \subset A\text{-}\mathrm{Mod}_{\mathcal{K}}$. See [Bal05, §3]. See Remark 2.14 for the functors $F_A : \mathcal{K} \rightleftarrows A\text{-}\mathrm{Mod}_{\mathcal{K}} : U_A$.

3.2. Lemma. *For every thick $\otimes$-ideal $\mathcal{J}_0 \subseteq \mathcal{K}$, the thick subcategory $U_A^{-1}(\mathcal{J}_0)$ of $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ is $\otimes$-ideal and equals the thick $\otimes$-ideal $\langle F_A(\mathcal{J}_0) \rangle$ generated by $F_A(\mathcal{J}_0)$.*

Proof. Since A is separable, we have $z \leq F_A U_A(z)$ for every $z \in A\text{-}\mathrm{Mod}_{\mathcal{K}}$ (Remark 2.14). Now, if $y \in U_A^{-1}(\mathcal{J}_0)$ and $x \in \mathcal{K}$ then $U_A(y) \in \mathcal{J}_0$ and since $\mathcal{J}_0$ is $\otimes$-ideal, $\mathcal{J}_0 \ni x \otimes U_A(y) \simeq U_A(F_A(x) \otimes_A y)$. So $F_A(x) \otimes_A y \in U_A^{-1}(\mathcal{J}_0)$. Hence for every $z \in A\text{-}\mathrm{Mod}_{\mathcal{K}}$, we have $U_A^{-1}(\mathcal{J}_0) \ni F_A(U_A(z)) \otimes_A y \geq z \otimes_A y$ hence $z \otimes_A y \in U_A^{-1}(\mathcal{J}_0)$ as wanted. So, $U_A^{-1}(\mathcal{J}_0)$ is $\otimes$-ideal. Let us show that $U_A^{-1}(\mathcal{J}_0)$ is the smallest thick $\otimes$-ideal containing $F_A(\mathcal{J}_0)$. On the one hand, $U_A F_A(x) \simeq A \otimes x$ belongs to $\mathcal{J}_0$ for every $x \in \mathcal{J}_0$. So, $F_A(\mathcal{J}_0) \subseteq U_A^{-1}(\mathcal{J}_0)$ and therefore $\langle F_A(\mathcal{J}_0) \rangle \subseteq U_A^{-1}(\mathcal{J}_0)$ by the above discussion. Conversely, if $z \in U_A^{-1}(\mathcal{J}_0)$ then $F_A(U_A(z)) \in F_A(\mathcal{J}_0)$ belongs to $\langle F_A(\mathcal{J}_0) \rangle$, hence so does every direct summand of $F_A(U_A(z))$, like z itself. $\square$

3.3. Lemma. *Let $\mathcal{P} \in \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ and let $s \in A\text{-}\mathrm{Mod}_{\mathcal{K}}$ be an object such that $U_A(s) \in \varphi_A(\mathcal{P})$. Then $s \in \mathcal{P}$.*

Proof. $U_A(s) \in \varphi_A(\mathcal{P}) = F_A^{-1}(\mathcal{P})$ reads $\mathcal{P} \ni F_A(U_A(s)) \geq s$ and $\mathcal{P}$ is thick. $\square$

3.4. Theorem. *Let A be a tt-ring in a tt-category $\mathcal{K}$. Then:*

(a) *For every $x \in \mathcal{K}$, we have $\varphi_A^{-1}(\mathrm{supp}(x)) = \mathrm{supp}(F_A(x))$ in $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$.*

(b) *Let $S \subset A\text{-}\mathrm{Mod}_{\mathcal{K}}$ be a $\otimes$-multiplicative collection of objects and consider the closed subset $\mathcal{Z}(S) = \cap_{s \in S} \mathrm{supp}(s)$ in $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$. Then its image is closed too; more precisely $\varphi_A(\mathcal{Z}(S)) = \mathcal{Z}(U_A(S))$ where $U_A(S) = \left\{ U_A(s) \,\middle|\, s \in S \right\}$ (not necessarily $\otimes$-multiplicative). In particular, φ_A is a closed map.*

(c) *For all $y \in A\text{-}\mathrm{Mod}_{\mathcal{K}}$, we have $\varphi_A(\mathrm{supp}(y)) = \underset{n \geq 1}{\cap} \mathrm{supp}(U_A(y^{\otimes n}))$ in $\mathrm{Spc}(\mathcal{K})$.*
$(^1)$

(d) *The image of $\varphi_A : \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \to \mathrm{Spc}(\mathcal{K})$ is exactly the support of A.*

Proof. Part (a) is a general property of $\mathrm{Spc}(F)$ for any tt-functor F, see [Bal05, Prop. 3.6]. For (b), note that $\mathcal{Z}(S) = \left\{ \mathcal{P} \,\middle|\, S \cap \mathcal{P} = \varnothing \right\}$. It follows from Lemma 3.3 that $\varphi_A(\mathcal{Z}(S)) \subseteq \mathcal{Z}(U_A(S))$. Conversely, suppose that $\mathcal{P}_0 \in \mathrm{Spc}(\mathcal{K})$ is such that $U_A(s) \notin \mathcal{P}_0$ for all $s \in S$ (∗) and consider the thick $\otimes$-ideal $\mathcal{J} := U_A^{-1}(\mathcal{P}_0)$ in $A\text{-}\mathrm{Mod}_{\mathcal{K}}$ (Lemma 3.2). Consider also the $\otimes$-multiplicative subset $T := S \otimes_A \left\{ F_A(x) \,\middle|\, x \in \mathcal{K} \smallsetminus \mathcal{P}_0 \right\}$ in $A\text{-}\mathrm{Mod}_{\mathcal{K}}$. We claim that $T \cap \mathcal{J} = \varnothing$. Indeed, suppose *ab absurdo*, that there exists $s \in S$ and $x \in \mathcal{K} \smallsetminus \mathcal{P}_0$ such that $s \otimes_A F_A(x) \in \mathcal{J} = U_A^{-1}(\mathcal{P}_0)$. Then $\mathcal{P}_0 \ni U_A(s \otimes_A F_A(x)) \simeq U_A(s) \otimes x$. Since $x \notin \mathcal{P}_0$ and $\mathcal{P}_0$ is prime, this implies

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^{1} When $\mathcal{K}$ is rigid, or A has finite degree, one can prove $\varphi_A(\mathrm{supp}(y)) = \mathrm{supp}(U_A(y))$.

that $U_A(s) \in \mathcal{P}_0$ which contradicts $(*)$. By the Existence Lemma 2.8, there exists a prime $\mathcal{P} \subset A\text{-}\mathrm{Mod}_{\mathcal{K}}$ such that $\mathcal{P} \supseteq \mathcal{J}$ and $\mathcal{P} \cap T = \varnothing$. These properties imply

$$(1)\ \mathcal{P} \supseteq U_A^{-1}(\mathcal{P}_0), \qquad (2)\ \mathcal{P} \cap S = \varnothing \qquad (3)\ \mathcal{P} \cap F_A(\mathcal{K} \smallsetminus \mathcal{P}_0) = \varnothing.$$

Since $U_A^{-1}(\mathcal{P}_0) = \langle F_A(\mathcal{P}_0) \rangle$ by Lemma 3.2, the first relation above implies $\mathcal{P}_0 \subseteq F_A^{-1}(\mathcal{P})$. Combining with (3), which implies $F_A^{-1}(\mathcal{P}) \subseteq \mathcal{P}_0$, we get $\mathcal{P}_0 = F_A^{-1}(\mathcal{P}) = \varphi_A(\mathcal{P})$. Finally, (2) reads $\mathcal{P} \in \mathcal{Z}(S)$ hence $\mathcal{P}_0 = \varphi_A(\mathcal{P}) \in \varphi_A(\mathcal{Z}(S))$ as wanted.

For (c), let $S = \{y^{\otimes n}, n \geq 1\}$ so that $\mathcal{Z}(S) = \mathrm{supp}(y)$. By (b), we deduce $\varphi_A(\mathrm{supp}(y)) = \cap_{n \geq 1} \mathrm{supp}(U_A(y^{\otimes n}))$.

Part (d) follows from (c) applied to $y = \mathbb{1}_A$ since $U_A(\mathbb{1}_A) = A$. $\qquad \square$

3.5. Theorem (Going-Up). *Let A be a tt-ring in $\mathcal{K}$. Let $\mathcal{Q} \in \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ be a point and $\mathcal{P} := \varphi_A(\mathcal{Q})$ its image in $\mathrm{Spc}(\mathcal{K})$. Let $\mathcal{P}' \in \overline{\{\mathcal{P}\}}$ be a point in the closure of $\mathcal{P}$. Then there exist $\mathcal{Q}' \in \overline{\{\mathcal{Q}\}}$ in the closure of $\mathcal{Q}$ such that $\varphi_A(\mathcal{Q}') = \mathcal{P}'$. In cash: if $\mathcal{P}' \subseteq \mathcal{P} = F_A^{-1}(\mathcal{Q})$ then there exists $\mathcal{Q}' \subseteq \mathcal{Q}$ such that $F_A^{-1}(\mathcal{Q}') = \mathcal{P}'$.*

Proof. This follows by Theorem 3.4 (b) with $S = (A\text{-}\mathrm{Mod}_{\mathcal{K}}) \smallsetminus \mathcal{Q}$. Indeed, $\overline{\{\mathcal{Q}\}} = \mathcal{Z}(S)$ for that particular S (see [Bal05, Prop. 2.9]) and $\mathcal{P}' \in \mathcal{Z}(U_A(S))$ since for every $s \notin \mathcal{Q}$, Lemma 3.3 implies that $U_A(s) \notin \varphi_A(\mathcal{Q}) = \mathcal{P} \supseteq \mathcal{P}'$ so $U_A(s) \notin \mathcal{P}'$. $\qquad \square$

3.6. *Remark.* Let $F : \mathcal{K} \to \mathcal{L}$ be a tt-functor, A a tt-ring in $\mathcal{K}$ and $B := F(A)$ the image tt-ring in $\mathcal{L}$. Then there exists a tt-functor $\bar{F} : A\text{-}\mathrm{Mod}_{\mathcal{K}} \longrightarrow B\text{-}\mathrm{Mod}_{\mathcal{L}}$ such that $\bar{F} F_A = F_B F$ and $U_B \bar{F} = F U_A$ (see [Bal14, Rem. 1.6]) :

$$(3.7) \qquad \begin{array}{ccc} \mathcal{K} & \xrightarrow{\ F\ } & \mathcal{L} \\ {\scriptstyle F_A}\Big\updownarrow{\scriptstyle U_A} & & {\scriptstyle F_B}\Big\updownarrow{\scriptstyle U_B} \\ A\text{-}\mathrm{Mod}_{\mathcal{K}} & \xrightarrow{\ \bar{F}\ } & B\text{-}\mathrm{Mod}_{\mathcal{L}} \end{array}$$

In two places below, we shall apply this construction to $F = F_A$ itself. To avoid confusion, let us say that A_1, A_2 are two tt-rings in $\mathcal{K}$ (later to be $A_1 = A_2 = A$). In the above notation, we set $A = A_2$, $\mathcal{L} = A_1\text{-}\mathrm{Mod}_{\mathcal{K}}$ and $F = F_{A_1}$. So the ring $B = F_{A_1}(A_2)$ in $\mathcal{L}$ has underlying ring $A_1 \otimes A_2$ in $\mathcal{K}$. By [Bal14, Rems. 1.4 and 1.5], we have an identification $B\text{-}\mathrm{Mod}_{\mathcal{L}} \cong (A_1 \otimes A_2)\text{-}\mathrm{Mod}_{\mathcal{K}}$ under which the two functors F_B and $\bar{F}$ become the two functors $A_1\text{-}\mathrm{Mod}_{\mathcal{K}} \to (A_1 \otimes A_2)\text{-}\mathrm{Mod}_{\mathcal{K}}$ and $A_2\text{-}\mathrm{Mod}_{\mathcal{K}} \to (A_1 \otimes A_2)\text{-}\mathrm{Mod}_{\mathcal{K}}$ associated to the two homomorphisms $1 \otimes \eta_2 : A_1 \to A_1 \otimes A_2$ and $\eta_1 \otimes 1 : A_2 \to A_1 \otimes A_2$.

3.8. Lemma. *Let $F : \mathcal{K} \to \mathcal{L}$ be a tt-functor, A a tt-ring in $\mathcal{K}$ and $B = F(A)$ as in Remark 3.6. Then diagram (3.7) yields a commutative square of spectra:*

$$(3.9) \qquad \begin{array}{ccccc} \mathrm{Spc}(\mathcal{K}) & \xleftarrow{\ \varphi=\mathrm{Spc}(F)\ } & \mathrm{Spc}(\mathcal{L}) & \ni & \mathcal{Q} \\ {\scriptstyle \varphi_A=\mathrm{Spc}(F_A)}\Big\uparrow & & \Big\uparrow{\scriptstyle \varphi_B=\mathrm{Spc}(F_B)} & & \\ \mathcal{P} \quad \in \quad \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) & \xleftarrow{\ \bar{\varphi}=\mathrm{Spc}(\bar{F})\ } & \mathrm{Spc}(B\text{-}\mathrm{Mod}_{\mathcal{L}}). & & \end{array}$$

Let $\mathcal{P} \in \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ and $\mathcal{Q} \in \mathrm{Spc}(\mathcal{L})$ such that $\varphi_A(\mathcal{P}) = \varphi(\mathcal{Q})$. Then there exists $\mathcal{Q}' \in \mathrm{Spc}(B\text{-}\mathrm{Mod}_{\mathcal{L}})$ such that $\bar{\varphi}(\mathcal{Q}') \in \overline{\{\mathcal{P}\}}$ (i.e. $\bar{\varphi}(\mathcal{Q}') \subseteq \mathcal{P}$) and $\varphi_B(\mathcal{Q}') = \mathcal{Q}$.

Proof. Consider the $\otimes$-multiplicative $S \subset B\text{-}\mathrm{Mod}_{\mathcal{L}}$ given by $S = \big\{ \bar{F}(s) \,\big|\, s \in A\text{-}\mathrm{Mod}_{\mathcal{K}}, \ s \notin \mathcal{P} \big\}$. We claim that $\mathcal{Q} \cap U_B(S) = \varnothing$. Indeed, if $\mathcal{Q} \ni U_B(\bar{F}(s)) \simeq F(U_A(s))$ for some $s \notin \mathcal{P}$ then $U_A(s) \in F^{-1}(\mathcal{Q}) = \varphi(\mathcal{Q}) = \varphi_A(\mathcal{P})$ hence $s \in \mathcal{P}$ by Lemma 3.3, which contradicts the choice of s. In other words, $\mathcal{Q} \in \mathcal{Z}(U_B(S))$ and by
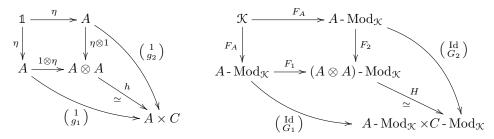
Theorem 3.4 (b) applied to the tt-ring B in the tt-category $\mathcal{L}$, there exists $\mathcal{Q}' \in \mathcal{Z}(S)$ such that $\varphi_B(\mathcal{Q}') = \mathcal{Q}$. The property $\mathcal{Q}' \in \mathcal{Z}(S)$ means that $\bar{F}((A\text{-}\mathrm{Mod}_{\mathcal{K}})-\mathcal{P})\cap\mathcal{Q}' = \varnothing$ which means that $\bar{\varphi}(\mathcal{Q}') = \bar{F}^{-1}(\mathcal{Q}') \subseteq \mathcal{P}$, as claimed.   □

3.10. **Theorem.** *Let $A \in \mathcal{K}$ be a tt-ring of finite degree $d \geq 1$ (see Remark 2.18).*
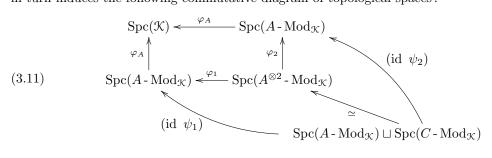
(a) *Incomparability: Let $\mathcal{P}, \mathcal{P}' \in \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ be two primes such that $\mathcal{P} \subseteq \mathcal{P}'$ (i.e. $\mathcal{P} \in \overline{\{\mathcal{P}'\}}$) and $\varphi_A(\mathcal{P}) = \varphi_A(\mathcal{P}')$. Then $\mathcal{P} = \mathcal{P}'$.*

(b) *The space $\mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ has the same Krull dimension as $\mathrm{supp}(A)$ – the latter as a subspace of $\mathrm{Spc}(\mathcal{K})$.*

(c) *The fibers of the map $\varphi_A : \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \to \mathrm{Spc}(\mathcal{K})$ are discrete (have Krull dimension zero) and contain at most d points.*

Proof. Note right away that (b) follows from part (a), Theorem 3.4 (d) and Going-Up Theorem 3.5. Similarly, (a) implies the discretion of the fibers in (c). So let us prove Incomparability and the bound on the cardinality of the fibers by induction on $d = \deg(A)$. For $d = 0$, we have $A = \mathbb{1}$ and there is nothing to prove. This being essentially a convention, let us discuss the next case. For $d = 1$, we use [Bal14, Prop. 4.1] which tells us that the multiplication $\mu : A \otimes A \overset{\sim}{\to} A$ is an isomorphism and F_A is simply a (Bousfield, smashing) localization, in such a way that the map $\varphi_A : \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \to \mathrm{Spc}(\mathcal{K})$ is the inclusion of an open and closed component of $\mathrm{Spc}(\mathcal{K})$ corresponding to $\mathrm{supp}(A)$; again in this case the results are straightforward. Suppose $\deg(A) > 1$ and the result true for tt-rings of degree $d-1$.

Let us do a preparation useful for (a) and (c). By Theorem 2.19, there exists a ring isomorphism $h = \binom{\mu}{g} : A \otimes A \overset{\sim}{\to} A \times C$ such that C has degree $d - 1$, when viewed as tt-ring in $A\text{-}\mathrm{Mod}_{\mathcal{K}}$. To be careful, let us write $\bar{C} \in A\text{-}\mathrm{Mod}_{\mathcal{K}}$ for C viewed as an A-module via $g\,(1 \otimes \eta)$, so that $C = U_A(\bar{C})$. See [Bal14, Rem. 1.5], which also recalls the canonical equivalence $\bar{C}\text{-}\mathrm{Mod}_{\mathcal{L}} \cong C\text{-}\mathrm{Mod}_{\mathcal{K}}$ where $\mathcal{L} := A\text{-}\mathrm{Mod}_{\mathcal{K}}$ in such a way that $F_{\bar{C}} \circ F_A \cong F_C$. Consider now the left-hand diagram of tt-rings below (defining the two homomorphisms $g_i : A \to C$ for $i = 1, 2$):

These homomorphisms induce the above right-hand diagram of tt-functors, which in turn induces the following commutative diagram of topological spaces:

(3.11)

where $\varphi_i = \mathrm{Spc}(F_i)$ and $\psi_i = \mathrm{Spc}(G_i)$, for $i = 1, 2$. Note that the upper-left square in (3.11) is nothing but (3.9) applied to $F = F_A$ itself. (See the explanations in the second half of Remark 3.6.) By the construction of C and the discussion above, the functor $G_1 : A\text{-}\mathrm{Mod}_{\mathcal{K}} \to C\text{-}\mathrm{Mod}_{\mathcal{K}} \cong \bar{C}\text{-}\mathrm{Mod}_{\mathcal{L}}$ is just an extension-of-scalar $F_{\bar{C}}$ with respect to the tt-ring $\bar{C}$ in $A\text{-}\mathrm{Mod}_{\mathcal{K}}$, which has degree $d-1$. We can therefore apply the induction hypothesis to the map ψ_1, i.e. we can assume that (a) and (c) hold for ψ_1.

Before jumping into the proof of (a), note that since $\varphi_A = F_A^{-1}(-)$ commutes with arbitrary intersection, an easy application of Zorn's Lemma shows that any φ_A-fiber admits a minimal prime inside any given $\mathcal{P}$. So, we can assume $\mathcal{P}$ minimal in its φ_A-fiber: If $\tilde{\mathcal{P}} \subseteq \mathcal{P}$ and $\varphi_A(\tilde{\mathcal{P}}) = \varphi_A(\mathcal{P})$ then $\tilde{\mathcal{P}} = \mathcal{P}$. By Lemma 3.8, there exists $\mathcal{Q}' \in \mathrm{Spc}(A^{\otimes 2}\text{-}\mathrm{Mod}_{\mathcal{K}})$ such that $\varphi_1(\mathcal{Q}') \in \overline{\{\mathcal{P}\}}$ and $\varphi_2(\mathcal{Q}') = \mathcal{P}'$. In the decomposition of $\mathrm{Spc}(A^{\otimes 2})$ as $\mathrm{Spc}(A) \sqcup \mathrm{Spc}(C)$, we have two possibilities. Either $\mathcal{Q}' \in \mathrm{Spc}(A)$, in which case (since $\varphi_1 = \varphi_2 = \mathrm{id}$ on the $\mathrm{Spc}(A)$ part) we get $\mathcal{P}' = \varphi_2(\mathcal{Q}') = \mathcal{Q}' = \varphi_1(\mathcal{Q}') \in \overline{\{\mathcal{P}\}}$, meaning $\mathcal{P}' \subseteq \mathcal{P}$, and we have $\mathcal{P} = \mathcal{P}'$ as wanted. Or $\mathcal{Q}' \in \mathrm{Spc}(C)$. In that case, by Going-Up Theorem 3.5 applied to φ_2 (i.e. to extension-of-scalar F_2), we deduce from $\varphi_2(\mathcal{Q}') = \mathcal{P}' \supseteq \mathcal{P}$ the existence of $\mathcal{Q} \subseteq \mathcal{Q}'$ such that $\varphi_2(\mathcal{Q}) = \mathcal{P}$. Since $\mathcal{Q}$ belongs to $\overline{\{\mathcal{Q}'\}}$, it is also in the $\mathrm{Spc}(C)$ part. Now $\varphi_1(\mathcal{Q}) \subseteq \varphi_1(\mathcal{Q}') \subseteq \mathcal{P}$ and since $\varphi_A\varphi_1 = \varphi_A\varphi_2$, the point $\varphi_1(\mathcal{Q})$ belongs to the same φ_A-fiber as $\mathcal{P}$. Since $\mathcal{P}$ is minimal in its fiber, we must have $\varphi_1(\mathcal{Q}) = \varphi_1(\mathcal{Q}') = \mathcal{P}$. In short, we have $\mathcal{Q} \subseteq \mathcal{Q}'$ in $\mathrm{Spc}(C)$ such that $\psi_1(\mathcal{Q}) = \psi_1(\mathcal{Q}')$. As explained above, we can apply the induction hypothesis to ψ_1, hence deduce $\mathcal{Q} = \mathcal{Q}'$. Taking the image under φ_2 finally gives $\mathcal{P} = \mathcal{P}'$, as wanted.

For (c), let $\mathcal{P}_0 \in \mathrm{Spc}(\mathcal{K})$ and consider its fiber $\varphi_A^{-1}(\mathcal{P}_0) \subseteq \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$. We need to show that $|\varphi_A^{-1}(\mathcal{P}_0)| \leq d$. If it is empty, we are done. Otherwise, pick $\mathcal{P} \in \varphi_A^{-1}(\mathcal{P}_0)$. Consider Diagram 3.11 again. We are going to define an injection $\varphi_A^{-1}(\mathcal{P}_0) \smallsetminus \{\mathcal{P}\} \hookrightarrow \psi_1^{-1}(\mathcal{P}) \subseteq \mathrm{Spc}(C)$. This is sufficient since by induction hypothesis, the fibers of ψ_1 have at most $d-1$ points. For every $\mathcal{P}' \in \varphi_A^{-1}(\mathcal{P}_0)$ with $\mathcal{P}' \neq \mathcal{P}$, there exists by Lemma 3.8 a prime $\mathcal{Q}' \in \mathrm{Spc}(A^{\otimes 2}\text{-}\mathrm{Mod}_{\mathcal{K}})$ such $\varphi_2(\mathcal{Q}') = \mathcal{P}'$ and $\varphi_1(\mathcal{Q}') \subseteq \mathcal{P}$. Since $\varphi_A\varphi_1 = \varphi_A\varphi_2$, the prime $\varphi_1(\mathcal{Q}')$ goes to $\mathcal{P}_0$ under φ_A and therefore $\varphi_1(\mathcal{Q}') = \mathcal{P}$ by Incomparability (a). Under the decomposition of $\mathrm{Spc}(A^{\otimes 2})$ as $\mathrm{Spc}(A) \sqcup \mathrm{Spc}(C)$, we cannot have $\mathcal{Q}' \in \mathrm{Spc}(A)$ for then the relations $\mathcal{P} = \varphi_1(\mathcal{Q}')$ and $\mathcal{P}' = \varphi_2(\mathcal{Q}')$ force $\mathcal{P} = \mathcal{P}'$ (since $\varphi_1 = \mathrm{id}$ and $\varphi_2 = \mathrm{id}$ on the $\mathrm{Spc}(A)$ part). Hence $\mathcal{Q}'$ belongs to the $\mathrm{Spc}(C)$ part and $\varphi_1(\mathcal{Q}') = \mathcal{P}$ now reads $\psi_1(\mathcal{Q}') = \mathcal{P}$. Therefore, we have constructed for every $\mathcal{P}' \in \varphi_A^{-1}(\mathcal{P}_0) \smallsetminus \{\mathcal{P}\}$ a prime $\mathcal{Q}' \in \psi_1^{-1}(\mathcal{P})$ such that $\psi_2(\mathcal{Q}') = \mathcal{P}'$. The latter relation shows that different $\mathcal{P}'$ have different $\mathcal{Q}'$. Hence the wanted injection $\varphi_A^{-1}(\mathcal{P}_0) \smallsetminus \{\mathcal{P}\} \hookrightarrow \psi_1^{-1}(\mathcal{P})$. $\qquad\square$

3.12. **Corollary.** *Let $A \in \mathcal{K}$ be a tt-ring of finite degree and let $F : \mathcal{K} \to \mathcal{L}$ be a tt-functor. Let $B = F(A)$ in $\mathcal{L}$. Then the induced commutative square (3.9)*

$$
\begin{array}{ccccc}
\mathrm{Spc}(\mathcal{K}) & \xleftarrow{\ \varphi\ } & \mathrm{Spc}(\mathcal{L}) & \ni & \mathcal{Q} \\
{\scriptstyle \varphi_A}\big\uparrow & & \big\uparrow{\scriptstyle \varphi_B} & & \\
\mathcal{P} \quad \in \quad \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) & \xleftarrow{\ \bar{\varphi}\ } & \mathrm{Spc}(B\text{-}\mathrm{Mod}_{\mathcal{L}}) & &
\end{array}
$$

is weakly cartesian *in the following sense: For every $\mathcal{P} \in \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}})$ and $\mathcal{Q} \in \mathrm{Spc}(\mathcal{L})$ such that $\varphi_A(\mathcal{P}) = \varphi(\mathcal{Q})$ there exists $\mathcal{Q}' \in \mathrm{Spc}(B\text{-}\mathrm{Mod}_{\mathcal{L}})$ such that $\bar{\varphi}(\mathcal{Q}') = \mathcal{P}$ and $\varphi_B(\mathcal{Q}') = \mathcal{Q}$.*

Proof. Choose $\mathcal{Q}'$ as in Lemma 3.8, i.e. with $\bar{\varphi}(\mathcal{Q}') \subseteq \mathcal{P}$ and $\varphi_B(\mathcal{Q}') = \mathcal{Q}$. Then $\varphi_A(\bar{\varphi}(\mathcal{Q}')) = \varphi(\varphi_B(\mathcal{Q}')) = \varphi(\mathcal{Q}) = \varphi_A(\mathcal{P})$. So $\bar{\varphi}(\mathcal{Q}') = \mathcal{P}$ by Theorem 3.10 (a). $\square$

3.13. *Remark.* The above square is not cartesian in general. For instance, using $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$, we see that for $\mathcal{K} = D^b(\mathbb{R})$ and $A = \mathbb{C}$, and thus $\mathcal{L} = D^b(\mathbb{C})$, the square has one point in three corners but two points in the bottom-right corner.

3.14. **Theorem.** *Let A be a tt-ring of finite degree. Then we have a coequalizer*

$$\mathrm{Spc}(A^{\otimes 2}\text{-}\mathrm{Mod}_{\mathcal{K}}) \xrightarrow[\varphi_2]{\varphi_1} \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \xrightarrow{\varphi_A} \mathrm{supp}(A)$$

where φ_i is the map induced by extension-of-scalars $F_i : A\text{-}\mathrm{Mod}_{\mathcal{K}} \longrightarrow A^{\otimes 2}\text{-}\mathrm{Mod}_{\mathcal{K}}$ along the two obvious homomorphisms $f_i : A \to A \otimes A$, i.e. $f_1 = 1 \otimes \eta$ and $f_2 = \eta \otimes 1$.

Proof. Let W be the coequalizer of φ_1 and φ_2. Since $\varphi_A \varphi_1 = \varphi_A \varphi_2$, the map φ_A induces a continuous map $\bar{\varphi} : W \to \mathrm{supp}(A)$. By Theorem 3.4 (d), the map $\bar{\varphi}$ is surjective. It is injective by Corollary 3.12 for $F = F_A$. (See the explanations in the second half of Remark 3.6 again.) Finally $\bar{\varphi}$ is a closed map since $\varphi_A : \mathrm{Spc}(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \to \mathrm{Spc}(\mathcal{K})$ is a closed map by Theorem 3.4 (b). $\square$

We now want to discuss conditions to replace the above $\mathrm{supp}(A)$ by $\mathrm{Spc}(\mathcal{K})$.

3.15. **Proposition.** *Let A be an object in $\mathcal{K}$. The following are equivalent:*

 (i) *It has maximal support: $\mathrm{supp}(A) = \mathrm{Spc}(\mathcal{K})$.*

 (ii) *The thick $\otimes$-ideal generated by A is the whole $\mathcal{K}$; in symbols: $\langle A \rangle = \mathcal{K}$.*

If A is a (separable) ring object then the above properties are also equivalent to:

 (iii) *In some (hence in every) distinguished triangle $J \xrightarrow{\xi} \mathbb{1} \xrightarrow{\eta} A \xrightarrow{\zeta} \Sigma J$ over the unit map η, the morphism ξ is $\otimes$-nilpotent.*

 (iv) *Extension-of-scalars $F_A : \mathcal{K} \to A\text{-}\mathrm{Mod}_{\mathcal{K}}$ is nil-faithful: If a morphism f in $\mathcal{K}$ is such that $F_A(f) = 0$ (i.e. $A \otimes f = 0$) then f is $\otimes$-nilpotent.*

 (v) *The image of the forgetful functor $U_A : A\text{-}\mathrm{Mod}_{\mathcal{K}} \to \mathcal{K}$ generates $\mathcal{K}$ as a thick $\otimes$-ideal; in symbols: $\langle U_A(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \rangle = \mathcal{K}$.*

Proof. (i) $\Longleftrightarrow$ (ii) is standard, see [Bal05, Cor. 2.5]. (ii)$\Rightarrow$(iii) follows from the observation that $\{x \in \mathcal{K} \mid \xi \otimes x \text{ is } \otimes\text{-nilpotent}\}$ is a thick $\otimes$-ideal of $\mathcal{K}$ which contains A since $\eta \otimes A$ is a split monomorphism. (iii)$\Rightarrow$(iv): If $f : x \to y$ is such that $A \otimes f = 0$ then $(\eta \otimes y)f = 0$ and therefore f factors through $\xi \otimes y$ which is $\otimes$-nilpotent. (iv)$\Rightarrow$(iii) is clear. (iii)$\Rightarrow$(ii) is well-known: by the Octahedron axiom $\mathrm{cone}(\xi^{\otimes n}) \in \langle \mathrm{cone}(\xi) \rangle = \langle A \rangle$ and if $\xi^{\otimes n} = 0 : J^{\otimes n} \to \mathbb{1}^{\otimes n}$ then $\mathbb{1} \in \langle \mathrm{cone}(\xi^{\otimes n}) \rangle$. To show (ii) $\Longleftrightarrow$ (v), it suffices to note that $\langle U_A(A\text{-}\mathrm{Mod}_{\mathcal{K}}) \rangle = \langle A \rangle$ in general. Since $A = U_A(\mathbb{1}_A)$, only the inclusion $\subseteq$ requires verification. If (x, ϱ) is an A-module then $\varrho : A \otimes x \to x$ is split by $\eta \otimes x$, hence $\langle A \rangle \ni x = U_A(x, \varrho)$. $\square$

3.16. *Definition.* Let us say that an object $A \in \mathcal{K}$ is *nil-faithful* if for every morphism f in $\mathcal{K}$, we have that $A \otimes f = 0$ implies $f^{\otimes n} = 0$ for some $n \geq 0$. For a tt-ring A this is equivalent to any of the properties of Proposition 3.15, see (iv).

3.17. **Corollary.** *Let $F : \mathcal{K} \to \mathcal{L}$ be a tt-functor. If A is a nil-faithful tt-ring in $\mathcal{K}$ then so is $F(A)$ in $\mathcal{L}$.*

Proof. Condition (iii) in Proposition 3.15 is preserved by tt-functors. $\square$

3.18. *Remark.* This Corollary really uses the tensor. For instance, if A only generates $\mathcal{K}$ as (thick) triangulated category, it is not necessarily true of $F(A)$ in $\mathcal{L}$.

3.19. **Theorem** (Descent-up-to-nilpotence). *Let A be a* nil-faithful *tt-ring of finite degree. With notation of Theorem 3.14, we have a coequalizer of topological spaces:*

$$\operatorname{Spc}(A^{\otimes 2}\operatorname{-Mod}_{\mathcal{K}}) \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \operatorname{Spc}(A\operatorname{-Mod}_{\mathcal{K}}) \overset{\varphi_A}{\longrightarrow} \operatorname{Spc}(\mathcal{K}).$$

Proof. In Theorem 3.14, use that $\operatorname{supp}(A) = \operatorname{Spc}(\mathcal{K})$ by Proposition 3.15. $\square$

4. Applications

Let G be a finite group and $G-\mathsf{sets}$ the category of finite left G-sets. Let $\Bbbk$ be a field of positive characteristic p dividing $|G|$, typically just the prime field $\mathbb{F}_p = \mathbb{Z}/p$.

4.1. *Construction.* Let $\mathcal{C}_{\mathbb{Z}}(\mathbb{Z}G)$ be the category of those finitely generated left $\mathbb{Z}G$-modules which are free as $\mathbb{Z}$-modules. It is a tensor category with $\otimes_{\mathbb{Z}}$ and diagonal G-action. Consider for every $X \in G-\mathsf{sets}$ the permutation $\mathbb{Z}G$-module $\mathbb{Z}X$ and, following [Bal15], define on it a $\mathbb{Z}$-bilinear multiplication μ_X by setting $x \cdot x = x$ for all $x \in X$ and $x \cdot x' = 0$ when $x \neq x'$. This gives a ring object $\underline{A}(X) = (\mathbb{Z}X, \mu_X, \eta_X)$ in the category $\mathcal{C}_{\mathbb{Z}}(\mathbb{Z}G)$ with unit $\eta_X : \mathbb{Z} \to \mathbb{Z}X$ given by $1 \mapsto \sum_{x \in X} x$. For every G-map $f : X \to Y$, define $\underline{A}(f) : \underline{A}(Y) \to \underline{A}(X)$ by $y \mapsto \sum_{x \in f^{-1}(y)} x$. For instance, $\underline{A}(X \to *) = \eta_X$. The ring object $\underline{A}(X)$ is separable via $\sigma : \underline{A}(X) \to \underline{A}(X) \otimes \underline{A}(X)$ given by $x \mapsto x \otimes x$. (Note that this map does not come from $G-\mathsf{sets}$.)

Consequently, for every tensor functor $\Phi : \mathcal{C}_{\mathbb{Z}}(\mathbb{Z}G) \longrightarrow \mathcal{K}$ to a tt-category $\mathcal{K}$, for instance the standard $\mathcal{C}_{\mathbb{Z}}(\mathbb{Z}G) \longrightarrow \mathrm{D}^{\mathrm{b}}(\Bbbk G)$ or $\mathcal{C}_{\mathbb{Z}}(\mathbb{Z}G) \longrightarrow \operatorname{stab}(\Bbbk G)$, the composite functor $A := \Phi \circ \underline{A} : G-\mathsf{sets}^{\mathrm{op}} \longrightarrow \mathcal{K}$ provides tt-rings $A(X)$ in $\mathcal{K}$ and ring homomorphisms $A(f) : A(Y) \to A(X)$ for every $f : X \to Y$ in $G-\mathsf{sets}$.

4.2. *Example.* For $X = G/H$ an orbit, our tt-ring $A_H^G = \Bbbk(G/H)$ is such an $A(X)$.

4.3. **Theorem** (Serre's vanishing of Bocksteins, gone tt). *Consider as above $A :$ $G-\mathsf{sets}^{\mathrm{op}} \longrightarrow \mathrm{D}^{\mathrm{b}}(\Bbbk G)$ given by $X \mapsto A(X) = \Bbbk X$. Let[2] $\operatorname{Abelem}(G)$ be the collection of elementary abelian p-subgroups of G (or only the maximal ones, or representatives up to conjugacy). Then the following tt-ring in $\mathrm{D}^{\mathrm{b}}(\Bbbk G)$*

$$A_{elem} := \prod_{H \in \operatorname{Abelem}(G)} A(G/H)$$

is nil-faithful in the sense of Definition 3.16 (and Proposition 3.15).

Proof. Let $\mathcal{K} = \mathrm{D}^{\mathrm{b}}(\Bbbk G)$. Since $A(G/H) \otimes - \simeq \operatorname{Res}_H^G$ is faithful when $[G : H]$ is prime to p, we reduce to G a p-group. By induction on $|G|$ it suffices to show that for every p-group G which is not elementary abelian, the tt-ring

$$A := \prod_{H \subsetneqq G} A(G/H)$$

is nil-faithful in $\mathrm{D}^{\mathrm{b}}(\Bbbk G)$. By Proposition 3.15 (v) again, it suffices to prove that $\langle A \rangle = \mathrm{D}^{\mathrm{b}}(\Bbbk G)$. Now, Serre's Theorem [Ser65, Prop. 4] precisely says that there are proper subgroups (of index p) $H_1, \ldots, H_m$ of G such that the product $\beta(z_1) \cdots \beta(z_m)$

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[2]In French Picardie, "abelem" is a clever contraction for "abélien élémentaire".

vanishes in $\mathrm{H}^{2m}(G,\Bbbk)$, where $\beta(z_i) \in \mathrm{H}^2(G,\Bbbk)$ is the Bockstein element associated to $z_i : G \twoheadrightarrow G/H_i \simeq \mathbb{Z}/p$. As an element of $\mathrm{Ext}^2_{\Bbbk G}(\Bbbk,\Bbbk) \cong \mathrm{H}^2(G,\Bbbk)$, this Bockstein is given by an exact sequence of the form

$$(4.4) \qquad \beta(z_i) = \Big(0 \to \Bbbk \to \Bbbk(G/H_i) \to \Bbbk(G/H_i) \to \Bbbk \to 0 \Big).$$

If we also denote by $\beta(z_i) : \Bbbk \to \Bbbk[2]$ the corresponding element in $\mathrm{Hom}_{\mathcal{K}}(\Bbbk,\Bbbk[2]) \cong \mathrm{Ext}^2_{\Bbbk G}(\Bbbk,\Bbbk)$ then its cone is the complex $\cdots \to 0 \to \Bbbk(G/H_i) \to \Bbbk(G/H_i) \to 0 \to \cdots$ appearing above, "in the middle". In particular, $\mathrm{cone}(\beta(z_i)) \in \langle \Bbbk(G/H_i) \rangle = \langle A(G/H_i) \rangle$. Therefore, an easy application of the octahedron (or just Yoneda splice) shows that the cone of our product $\beta(z_1) \cdots \beta(z_m) = \beta(z_1) \otimes \cdots \otimes \beta(z_m)$ belongs to $\langle A(G/H_1), \ldots, A(G/H_m) \rangle \subset \langle A \rangle$. Since this product is zero and since $\mathrm{cone}(0 : \Bbbk \to \Bbbk[2m]) = \Bbbk[1] \oplus \Bbbk[2m]$ we get $\Bbbk \in \langle A \rangle$ as wanted. $\qquad\square$

4.5. *Remark.* I am thankful to Raphaël Rouquier for simplifying my earlier proof of Theorem 4.3, which involved the following, perhaps interesting, observation. The exact sequence (4.4) is the cone of an isomorphism $\psi_i : J_i \xrightarrow{\sim} J_i^*[-2]$ in $\mathrm{D}^{\mathrm{b}}(\Bbbk G)$, where $J_i = (\cdots 0 \to \Bbbk \to \Bbbk(G/H_i) \to 0 \cdots)$ fits in the distinguished triangle $J_i \xrightarrow{\xi_i} \Bbbk \xrightarrow{\eta_i} A(G/H_i) \to \Sigma J_i$ as in Proposition 3.15. A direct computation shows that the Bockstein $\beta(z_i)$ is equal to the composite $\xi_i[2] \circ \psi_i^{-1} \circ \xi_i^*$. In other words, our morphism ξ_i appears "twice" in the corresponding Bockstein.

The analogue of Theorem 4.3 also holds for $\mathcal{K} = \mathrm{stab}(\Bbbk G)$ by Corollary 3.17.

4.6. *Definition.* Recall that the *orbit category* $\mathrm{Or}(G)$ of the group G has objects indexed by subgroups $H \leq G$, thought of as the corresponding orbit, i.e. morphisms are given by $\mathrm{Mor}_{\mathrm{Or}(G)}(H,K) = \mathrm{Mor}_{G-\mathsf{sets}}(G/H, G/K)$. We write $\mathrm{Or}(G, \mathrm{Abelem})$ for the full subcategory of $\mathrm{Or}(G)$ on those H which are elementary abelian p-groups.

4.7. *Remark.* In [Car00], Carlson already noted that Serre's theorem implies that the support variety $\mathcal{V}_G = \mathrm{Proj}(\mathrm{H}^\bullet(G,\Bbbk))$ is covered by the images of the $\mathcal{V}_H$

$$(4.8) \qquad\qquad \mathcal{V}_G = \bigcup_{H \in \mathrm{Abelem}(G)} \mathrm{Res}^* \mathcal{V}_H.$$

This equality is part of Quillen's Stratification Theorem (0.1), which says that:

$$(4.9) \qquad\qquad \mathcal{V}_G \cong \underset{H \in \mathrm{Or}(G,\mathrm{Abelem})}{\mathrm{colim}} \mathcal{V}_H.$$

In terms of cohomology rings, (4.8) says that $\mathrm{H}^\bullet(G,\Bbbk) \longrightarrow \prod_H \mathrm{H}^\bullet(H,\Bbbk)$ has nilpotent kernel whereas the colimit-version (4.9) is saying moreover that elements in $\lim_H \mathrm{H}^\bullet(H,\Bbbk)$ have some p^r-power coming from $\mathrm{H}^\bullet(G,\Bbbk)$. See for instance [Ben98, Cor. II.5.6.4]. Another way to see how the colimit-version (4.9) is more informative than (4.8) is to note that (4.8) does not prevent $\mathcal{V}_G$ from being a point, for instance, whereas (4.9) allowed Quillen to compute the Krull dimension of $\mathcal{V}_G$.

We now want to show that descent-up-to-nilpotence (Theorem 3.19) yields a generalization of Quillen's Stratification Theorem, in its strong form (4.9).

Quillen's Stratification Theorem (4.9) is the case $\mathcal{K} = \mathrm{stab}(\Bbbk G)$ of the following:

4.10. **Theorem.** *Let $\mathcal{K}$ be a tt-category and let $\Phi : \mathrm{D}^{\mathrm{b}}(\mathbb{F}_p G) \longrightarrow \mathcal{K}$ be a tt-functor. For every subgroup $H \leq G$ consider the tt-ring $A_H = \Phi(\mathbb{F}_p(G/H))$ in $\mathcal{K}$ and define*

$\mathcal{K}(H) = A_H\text{-}\mathrm{Mod}_\mathcal{K}$ *the corresponding category of modules. Then the collection of tt-functors* $F_{A_H} : \mathcal{K} = \mathcal{K}(G) \longrightarrow \mathcal{K}(H)$ *induces a homeomorphism*

$$(4.11) \qquad\qquad \bar{\varphi} : \operatorname*{colim}_{H \in \mathrm{Or}(G,\mathrm{Abelem})} \mathrm{Spc}(\mathcal{K}(H)) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K}).$$

Proof. Following Construction 4.1, we have a functor $A(-) : G\text{-}\mathsf{sets}^{\mathrm{op}} \to \mathcal{K}$, taking values in tt-rings and homomorphisms and whose restriction to $\mathrm{Or}(G)$ recovers the ring $A(G/H) = \Phi(\mathbb{F}_p(G/H)) = A_H$ of the statement. Since $A_G = \Phi(\Bbbk) = \mathbb{1}_\mathcal{K}$, since G/G is the final G-set and since $\mathrm{Spc}(-)$ is contravariant, we indeed have a canonical continuous map $\bar{\varphi}$ as in (4.11) such that precomposed with the canonical map $\mathrm{Spc}(\mathcal{K}(H)) \to \mathrm{colim}_H \mathrm{Spc}(\mathcal{K}(H))$ gives $\mathrm{Spc}(F_{A_H})$. Consider the tt-ring $A := \prod_{H \in \mathrm{Abelem}(G)} A_H$, which is the image in $\mathcal{K}$ of the tt-ring A_{elem} of Serre's Theorem 4.3. Since A_{elem} was nil-faithful in $\mathrm{D}^{\mathrm{perf}}(\mathbb{F}_p G)$, Corollary 3.17 implies that A is also nil-faithful in $\mathcal{K}$. By descent-up-to-nilpotence (Theorem 3.19), we have a coequalizer of spaces

$$(4.12) \qquad \mathrm{Spc}(A^{\otimes 2}\text{-}\mathrm{Mod}_\mathcal{K}) \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \mathrm{Spc}(A\text{-}\mathrm{Mod}_\mathcal{K}) \xrightarrow{\varphi_A} \mathrm{Spc}(\mathcal{K})$$

where the maps φ_i are induced by $f_i : A \to A \otimes A$ given by $f_1 = 1 \otimes \eta$ and $f_2 = \eta \otimes 1$. Let us identify $A_H \otimes A_K = \Phi(\Bbbk(G/H)) \otimes \Phi(\Bbbk(G/K)) \cong \Phi(\Bbbk(G/H \times G/K)) = A(G/H \times G/K)$, so that the components $A_H \to A_H \otimes A_K$ and $A_K \to A_H \otimes A_K$ of f_1 and $f_2 : A \to A \otimes A$ are simply the images under $A : G\text{-}\mathsf{sets}^{\mathrm{op}} \to \mathcal{K}$ of the two projections $\mathrm{pr}_1 : G/H \times G/K \to G/H$ and $\mathrm{pr}_2 : G/H \times G/K \to G/K$, respectively.

In $G\text{-}\mathsf{sets}$, for any two subgroups $H, K \leq G$, we have a Mackey isomorphism

$$\coprod_{[g] \in H \backslash G / K} \beta_g : \coprod_{[g] \in H \backslash G / K} G/(H^g \cap K) \overset{\sim}{\to} G/H \times G/K$$

where the map $\beta_g : G/(H^g \cap K) \longrightarrow G/H \times G/K$ is $[x] \mapsto ([xg^{-1}], [x])$. As usual, this map is non-canonical since it involves the choice of $g \in G$ for each double class $[g] \in H \backslash G / K$. Still, we get an isomorphism (first in $\mathrm{D}^{\mathrm{b}}(\mathbb{F}_p G)$ and then in $\mathcal{K}$):

$$\prod_{\substack{H,K \in \mathrm{Abelem}(G) \\ [g] \in H \backslash G / K}} A(\beta_g) : \quad A \otimes A = \prod_{H,K \in \mathrm{Abelem}(G)} A_H \otimes A_K \xrightarrow{\sim} \prod_{\substack{H,K \in \mathrm{Abelem}(G) \\ [g] \in H \backslash G / K}} A_{H^g \cap K}.$$

Replacing the decompositions for A and $A \otimes A$ in (4.12), we obtain a coequalizer

$$\coprod_{\substack{H,K \in \mathrm{Abelem}(G) \\ [g] \in H \backslash G / K}} \mathrm{Spc}(A_{H^g \cap K}) \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \coprod_{H \in \mathrm{Abelem}(G)} \mathrm{Spc}(A_H) \xrightarrow{\coprod \varphi_{A_H}} \mathrm{Spc}(\mathcal{K})$$

where the maps $\psi_1 : \mathrm{Spc}(A_{H^g \cap K}) \to \mathrm{Spc}(A_H)$ and $\psi_2 : \mathrm{Spc}(A_{H^g \cap K}) \to \mathrm{Spc}(A_K)$ are induced by the composition of the above maps $f_i : A \rightrightarrows A \otimes A$ with the maps $A(\beta_g) : A_H \otimes A_K \to A_{H^g \cap K}$ on components. Since all maps come through $A : G\text{-}\mathsf{sets}^{\mathrm{op}} \to \mathcal{K}$, we can compute these compositions in $G\text{-}\mathsf{sets}$ already and get

$$\begin{array}{ccc}
& & G/H \\
& \overset{\mathrm{pr}_1\,\beta_g = \alpha_g}{\nearrow} & \\
& & \uparrow\scriptstyle{\mathrm{pr}_1} \\
G/(H^g \cap K) & \xrightarrow{\quad\beta_g\quad} & G/H \times G/K \\
& & \downarrow\scriptstyle{\mathrm{pr}_2} \\
& \underset{\mathrm{pr}_2\,\beta_g = \alpha_1}{\searrow} & \\
& & G/K
\end{array}$$

where $\alpha_g : G/L \to G/L'$ denotes $[x] \mapsto [xg^{-1}]$ whenever ${}^gL \le L'$. So, for every $H, K \le G$ and $g \in G$, we have

$$(4.13) \quad \psi_1\big|_{\mathrm{Spc}(A_{H^g \cap K})} = \mathrm{Spc}(A(\alpha_g)) \qquad \text{and} \qquad \psi_2\big|_{\mathrm{Spc}(A_{H^g \cap K})} = \mathrm{Spc}(A(\alpha_1)).$$

We can now compare our coequalizer with the colimit as follows:

$$\coprod_{\substack{H,K \in \mathrm{Abelem}(G) \\ [g] \in H \backslash G / K}} \mathrm{Spc}(A_{H^g \cap K}) \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} \coprod_{H \in \mathrm{Abelem}(G)} \mathrm{Spc}(A_H) \xrightarrow{\ \sqcup\, \varphi_{A_H}\ } \mathrm{Spc}(\mathcal{K})$$

where π is the quotient map and $\bar{\varphi}$ is the above continuous map such that $\bar{\varphi}\pi = \sqcup_H \varphi_{A_H}$. By (4.13), the two maps ψ_1 and ψ_2 are the images of maps α_g and α_1 in $\mathrm{Or}(G, \mathrm{Abelem})$ via the functor $H \mapsto \mathrm{Spc}(A_H)$. Therefore by the colimit property $\pi \circ \psi_1 = \pi \circ \psi_2$. Hence, by the coequalizer property, there is a continuous map $\bar{\pi} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{colim}_H \mathrm{Spc}(A_H)$ such that $\bar{\pi} \circ (\sqcup \varphi_{A_H}) = \pi$, as in the above diagram. This $\bar{\pi}$ is therefore the inverse of $\bar{\varphi}$ since π and $\sqcup \varphi_{A_H}$ are epimorphisms. $\qquad \square$

4.14. *Example.* Let X be a scheme over $\Bbbk$ on which our finite group G acts. Let $\mathcal{K}(G) = \mathrm{D}_G^{\mathrm{b}}(\mathrm{VB}_X)$ be the bounded derived category of G-equivariant vector bundles on X. Then it receives $\mathrm{D}^{\mathrm{b}}(\Bbbk G) = \mathrm{D}_G^{\mathrm{b}}(\mathrm{Spec}(\Bbbk))$ and $A_H\text{-}\mathrm{Mod}_{K(G)} \cong \mathrm{D}_H^{\mathrm{b}}(\mathrm{VB}_X)$ for every subgroup $H \le G$ as can be readily verified from the restriction-coinduction adjunction. Alternatively, see [BDS14, §5]. Theorem 4.10 tells us that the spectrum of $\mathrm{D}_G^{\mathrm{b}}(\mathrm{VB}_X)$ is the colimit of the spectra of $\mathrm{D}_H^{\mathrm{b}}(\mathrm{VB}_X)$ over H in $\mathrm{Or}(G, \mathrm{Abelem})$. Furthermore, the Krull dimension of $\mathrm{Spc}(\mathrm{D}_G^{\mathrm{b}}(\mathrm{VB}_X))$ is the maximum of the Krull dimensions of $\mathrm{Spc}(\mathrm{D}_H^{\mathrm{b}}(\mathrm{VB}_X))$ among the elementary abelian p-subgroups $H \le G$.

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