TENSOR TRIANGULAR CHOW GROUPS

PAUL BALMER

Abstract. We propose a definition of the Chow group of a rigid tensor triangulated category. The basic idea is to allow "generalized" cycles, with non-integral coefficients. The precise choice of relations is open to some fine-tuning.

Hypothesis 1. Let $\mathcal{K}$ be an essentially small tensor triangulated category. Let us assume that its triangular spectrum in the sense of [1], $\mathrm{Spc}(\mathcal{K}) = \big\{ \mathcal{P} \subset \mathcal{K} \,\big|\, \mathcal{P} \text{ is prime} \big\}$, is a *noetherian* topological space, *i.e.* that every open of $\mathrm{Spc}(\mathcal{K})$ is quasi-compact. Let us also assume that $\mathcal{K}$ is *rigid*, as explained in [4] (or [2], where this property was called *strongly closed*). These hypotheses allow us to use the techniques of filtration of $\mathcal{K}$ by (generalized) dimension of the support.

Definition 2. As in [2, Def. 3.1], let us consider $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ a *dimension function*, meaning that $\mathcal{P} \subseteq \mathcal{Q} \implies \dim(\mathcal{P}) \le \dim(\mathcal{Q})$, with equality in the finite range only if $\mathcal{P} = \mathcal{Q}$ (*i.e.* $\mathcal{P} \subseteq \mathcal{Q}$ and $\dim(\mathcal{P}) = \dim(\mathcal{Q}) \in \mathbb{Z}$ forces $\mathcal{P} = \mathcal{Q}$). Examples are the Krull dimension of $\overline{\{\mathcal{P}\}}$ in $\mathrm{Spc}(\mathcal{K})$, or the opposite of its Krull codimension. Assuming $\dim(-)$ is clear from the context, we shall use the notation

$$\mathrm{Spc}(\mathcal{K})_{(p)} := \big\{ \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,\big|\, \dim(\mathcal{P}) = p \big\}\,.$$

Remark 3. In my opinion, there is nothing conceptually remarkable about the free abelian group on $\mathrm{Spc}(\mathcal{K})_{(p)}$. Therefore I propose another definition of p-dimensional cycles. This requires some preparation.

Definition 4. Recall from [3, § 4] that a rigid tensor triangulated category $\mathcal{L}$ is called *local* if $a \otimes b = 0$ implies $a = 0$ or $b = 0$. Conceptually, this means that $\mathrm{Spc}(\mathcal{L})$ is a local space, *i.e.* that $\mathrm{Spc}(\mathcal{L})$ has a unique closed point $* := 0 \subset \mathcal{L}$, which is prime by assumption.

Example 5. For every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, the following tensor triangulated category is local in the above sense:

$$\mathcal{K}_{\mathcal{P}} := \big(\mathcal{K}/\mathcal{P}\big)^{\natural}$$

where $\mathcal{K}/\mathcal{P}$ denotes the Verdier quotient and $(-)^{\natural}$ the idempotent completion. We call $\mathcal{K}_{\mathcal{P}}$ the *local category at* $\mathcal{P}$. There is an obvious (localization) functor

$$q_{\mathcal{P}} \,:\, \mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{P} \hookrightarrow \mathcal{K}_{\mathcal{P}}$$

composed of localization and idempotent completion. (The category $\mathcal{K}_{\mathcal{P}}$ can also be understood as the strict filtered colimit of the $\mathcal{K}(U)$ over those open subsets $U \subseteq \mathrm{Spc}(\mathcal{K})$ which contain $\mathcal{P}$. See more in [4, § 2.2] if helpful.) We can identify $\mathrm{Spc}(\mathcal{K}_{\mathcal{P}})$ with the subspace $\big\{ \mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \,\big|\, \mathcal{P} \in \overline{\{\mathcal{Q}\}} \big\}$ of $\mathrm{Spc}(\mathcal{K})$, hence the space $\mathrm{Spc}(\mathcal{K}_{\mathcal{P}})$ remains noetherian.

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Definition 6. Assuming that $\mathcal{L}$ is local and that $\mathrm{Spc}(\mathcal{L})$ is noetherian, the open complement of the unique closed point $\{*\}$ in $\mathrm{Spc}(\mathcal{L})$ is quasi-compact, *i.e.* $\{*\}$ is a "Thomason (closed) subset". Under the classification of thick $\otimes$-ideals of $\mathcal{L}$, see [1], this one-point subset corresponds to the minimal non-zero thick $\otimes$-ideal

$$\mathrm{Min}(\mathcal{L}) := \mathcal{L}_{\{*\}} = \big\{\, a \in \mathcal{L} \,\big|\, \mathrm{supp}(a) \subseteq \{*\} \,\big\}\,.$$

These are the objects with minimal possible support (empty or a point).

Remark 7. Some comments are in order:

(1) This subcategory was called the subcategory of *finite-length* objects in [2] and denoted $\mathrm{FL}(\mathcal{L})$. As far as I know, there is no reason for objects of $\mathrm{Min}(\mathcal{L})$ to have finite-length (in the categorical sense that they admit a finite filtration with simple subquotients). The present notation, $\mathrm{Min}(\mathcal{L})$, is less biased towards commutative algebra and therefore probably preferable. It is however an interesting question to find some structure theorems about $\mathrm{Min}(\mathcal{L})$.

(2) As the previous comment suggests, if we take $\mathcal{L} = \mathrm{K^b}(R\!-\!\mathrm{proj})$ the category of perfect complexes for R noetherian and local, then $\mathcal{L}$ is local and $\mathrm{Min}(\mathcal{L})$ is the subcategory of perfect complexes with finite-length homology.

(3) One can of course consider $\mathrm{Min}(\mathcal{L})$ even if $*$ is not Thomason but in that case it would just be the zero subcategory $0 = \mathcal{L}_{\varnothing}$.

Definition 8. Let $p \in \mathbb{Z}$. We define the group of *generalized p-cycles* to be

$$\mathrm{Z}_p(\mathcal{K}) := \bigoplus_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(p)}} K_0\big(\,\mathrm{Min}(\mathcal{K}_{\mathcal{P}})\big)\,,$$

where K_0 is the Grothendieck K-group (the quotient of the monoid of isomorphism classes $[a]$ of objects under $\oplus$, by the submonoid of those $[a] + [\Sigma b] + [c]$ for which there exists a distinguished triangle $a \to b \to c \to \Sigma a$).

Out of nostalgia for usual cycles, a generalized p-cycle can be written $\sum_{\mathcal{P}} \lambda_{\mathcal{P}} \cdot \mathcal{P}$ or $\sum_{\mathcal{P}} \lambda_{\mathcal{P}} \cdot \overline{\{\mathcal{P}\}}$, for $\lambda_{\mathcal{P}} \in K_0\big(\,\mathrm{Min}(\mathcal{K}_{\mathcal{P}})\big)$. This is a purely notational choice. The non-trivial point is that we allow coefficients $\lambda_{\mathcal{P}}$ to live in other abelian groups than $\mathbb{Z}$, namely the Grothendieck groups of the minimal categories at every $\mathcal{P}$.

Example 9. Let X be a (topologically) noetherian scheme and $\mathcal{K} = \mathrm{D^{perf}}(X)$ the derived category of perfect complexes, whose spectrum $\mathrm{Spc}(\mathcal{K}) \cong X$ recovers the underlying space of X. Let $\dim(-)$ be the (opposite of the) Krull (co)dimension. Then we recover the usual p-dimensional (resp. $(-p)$-codimensional) cycles. Indeed, we have by Thomason that $\mathcal{K}_{\mathcal{P}} \cong \mathrm{K^b}(\mathcal{O}_{X,x}\!-\!\mathrm{proj})$ if $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ corresponds to $x \in X$. The reason why integral coefficients suffice over regular schemes is that the group homomorphism defined by alternate sum of length of homology groups

$$K_0\big(\,\mathrm{Min}(\mathrm{K^b}(\mathcal{O}_{X,x}\!-\!\mathrm{proj}))\big) \longrightarrow \mathbb{Z}\,,$$

is an isomorphism if X is regular (at x). However, in general, the left-hand group could be tricky, as discussed for instance in Roberts-Srinivas [6].

Now to the relations. There might be several definitions of relations. The most flexible and most obvious one is the following.

Definition 10. For a (specialization) closed subset $Y \subset \mathrm{Spc}(\mathcal{K})$, we set $\dim(Y) = \sup\big\{\, \dim(\mathcal{P}) \,\big|\, \mathcal{P} \in Y \,\big\}$ and consider the filtration $\cdots \subset \mathcal{K}_{(p)} \subset \mathcal{K}_{(p+1)} \subset \cdots \subset \mathcal{K}$ by dimension of support

$$\mathcal{K}_{(p)} := \big\{\, a \in \mathcal{K} \,\big|\, \dim(\mathrm{supp}(a)) \le p \,\big\}\,.$$

By [2, Thm. 3.24], localization induces an equivalence

$$(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural \xrightarrow{\sim} \coprod_{\mathcal{P}\in\mathrm{Spc}(\mathcal{K})_{(p)}} \mathrm{Min}(\mathcal{K}_\mathcal{P}) \tag{11}$$

and consequently $\mathrm{Z}_p(\mathcal{K}) \cong K_0\big((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\big)$. Note that this definition of $\mathrm{Z}_p(\mathcal{K})$ does not need $\mathrm{Spc}(\mathcal{K})$ being noetherian. It also allows the definition of the *p-boundaries* $\mathrm{B}_p(\mathcal{K})$ as the image in $\mathrm{Z}_p(\mathcal{K})$ of $\mathrm{Ker}\big(K_0(\mathcal{K}_{(p)}) \to K_0(\mathcal{K}_{(p+1)})\big)$. In other words we have the diagram with exact rows

$$\begin{tikzcd}
\mathrm{Ker}(\iota) \arrow[r, tail] \arrow[d, two heads] & K_0(\mathcal{K}_{(p)}) \arrow[r, "\iota"] \arrow[d, two heads] & K_0(\mathcal{K}_{(p+1)}) \\
\mathrm{B}_p(\mathcal{K}) \arrow[r, tail] & \mathrm{Z}_p(\mathcal{K}) \arrow[r, two heads] & \mathrm{CH}_p(\mathcal{K})
\end{tikzcd}$$

in which we define $\mathrm{CH}_p(\mathcal{K}) := \mathrm{Z}_p(\mathcal{K})/\,\mathrm{B}_p(\mathcal{K})$ to be the quotient of p-cycles by p-boundaries. These groups could be called the *(K-theoretic) Chow groups of p-cycles in $\mathcal{K}$*, with respect to the chosen dimension function dim.

Remark 12. The above $\mathrm{Ker}(\iota)$ is an *ad hoc* replacement for the maybe more natural image of $K_1(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)})$ by a connecting homomorphism. The reason for the above definition is that triangulated categories do not behave well with higher K-theory. However, with this definition, it is not too hard to check that $\mathrm{CH}_p(\mathcal{K}) = \mathrm{CH}_p(X)$ when X is a regular scheme and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$. See more in Klein [5].

It is however tempting to give another definition of p-boundaries, closer to the classical ideas of equivalence of p-cycles by means of divisors of rational functions on $(p+1)$-dimensional varieties. We need a preparation.

Lemma 13. *Let $a \in \mathcal{K}_{(p+1)}$ be an object with support of dimension at most $p+1$ and let $\gamma : a \overset{\sim}{\to} a$ be an automorphism in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$. Choose a fraction $a \xrightarrow{\alpha} b \xleftarrow{\beta} a$ in $\mathcal{K}_{(p+1)}$ representing γ, so that* $\mathrm{cone}(\alpha)$ *and* $\mathrm{cone}(\beta)$ *both belong to $\mathcal{K}_{(p)}$. Then the difference* $[\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)]$ *in $K_0(\mathcal{K}_{(p)})$ belongs to* $\mathrm{Ker}\big(\iota : K_0(\mathcal{K}_{(p)}) \to K_0(\mathcal{K}_{p+1})\big)$ *and is independent of the choice of α and β.*

Proof. This is an immediate verification: In $K_0(\mathcal{K}_{(p+1)})$, we have $[\mathrm{cone}(\alpha)] = [b] - [a] = [\mathrm{cone}(\beta)]$, hence the first statement. Independence on the choice of the fraction up to amplification by a morphism $s : b \to b'$ with cone in $\mathcal{K}_{(p)}$ follows by the octahedron axiom: $[\mathrm{cone}(s\alpha)] = [\mathrm{cone}(s)] + [\mathrm{cone}(\alpha)]$ and $[\mathrm{cone}(s\beta)] = [\mathrm{cone}(s)] + [\mathrm{cone}(\beta)]$, so $[\mathrm{cone}(s\alpha)] - [\mathrm{cone}(s\beta)] = [\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)]$. ☐

Definition 14. Let $a \in \mathcal{K}_{(p+1)}$ and let $\gamma : a \overset{\sim}{\to} a$ be an automorphism in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$. Choose a fraction $a \xrightarrow{\alpha} b \xleftarrow{\beta} a$ in $\mathcal{K}_{(p+1)}$ representing γ, and let

$$\mathrm{div}(a \overset{\gamma}{\to} a) = [q(\mathrm{cone}(\alpha))] - [q(\mathrm{cone}(\beta))] \in \mathrm{B}_p(\mathcal{K})$$

where $q : \mathcal{K}_{(p)} \to (\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural$ is the canonical functor. We might call this element the *divisor* of $\gamma : a \overset{\sim}{\to} a$. This generalized p-cycle is a p-boundary by construction.

Remark 15. Of course, in view of the equivalence (11), we can also write

$$\mathrm{div}(\gamma) = \sum_{\mathcal{P}\in\mathrm{Spc}(\mathcal{K})_{(p)}} [q_\mathcal{P}(\mathrm{cone}(\alpha))] - [q_\mathcal{P}(\mathrm{cone}(\beta))]$$

where $q_{\mathcal{P}} : \mathcal{K} \longrightarrow \mathcal{K}_{\mathcal{P}}$ is the localization and where $\gamma = (a \xrightarrow{\alpha} b \xleftarrow{\beta} a)$ as before. The above formula for the divisor might look more familiar to the reader.

Remark 16. A priori, there might be more p-boundaries than the ones coming from the above divisors $\operatorname{div}(\gamma)$. This means that one might have a different Chow group $\mathrm{CH}'_p(\mathcal{K})$ defined as the quotient of $\mathrm{Z}_p(\mathcal{K})$ by the subgroup generated by those $\operatorname{div}(\gamma)$. This group $\mathrm{CH}'_p(\mathcal{K})$ would surject the group $\mathrm{CH}_p(\mathcal{K})$ of Definition 10. However, in the case of $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ for a (nice) regular scheme X, it might well be that CH'_p coincides with CH_p because all relations coming from K_1 seem to be captured by divisors. This point requires further investigation and we refer the interested reader to the forthcoming [5].

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Paul Balmer, Mathematics Department, UCLA, Los Angeles, CA 90095-1555, USA
E-mail address: balmer@math.ucla.edu
URL: http://www.math.ucla.edu/~balmer