TENSOR TRIANGULAR CHOW GROUPS

PAUL BALMER

Abstract. We propose a definition of the Chow group of a rigid tensor triangulated category. The basic idea is to allow "generalized" cycles, with non-integral coefficients. The precise choice of relations is open to some fine-tuning.

Hypothesis 1. Let \( \mathcal{K} \) be an essentially small tensor triangulated category. Let us assume that its triangular spectrum in the sense of [1], \( \text{Spc}(\mathcal{K}) = \{ \mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is prime} \} \), is a noetherian topological space, i.e. that every open of \( \text{Spc}(\mathcal{K}) \) is quasi-compact. Let us also assume that \( \mathcal{K} \) is rigid, as explained in [4] (or [2], where this property was called strongly closed). These hypotheses allow us to use the techniques of filtration of \( \mathcal{K} \) by (generalized) dimension of the support.

Definition 2. As in [2, Def. 3.1], let us consider \( \text{dim} : \text{Spc}(\mathcal{K}) \to \mathbb{Z} \cup \{ \pm \infty \} \) a dimension function, meaning that \( \mathcal{P} \subseteq \mathcal{Q} \Rightarrow \text{dim}(\mathcal{P}) \leq \text{dim}(\mathcal{Q}) \), with equality in the finite range only if \( \mathcal{P} = \mathcal{Q} \) (i.e. \( \mathcal{P} \subseteq \mathcal{Q} \) and \( \text{dim}(\mathcal{P}) = \text{dim}(\mathcal{Q}) \in \mathbb{Z} \) forces \( \mathcal{P} = \mathcal{Q} \)). Examples are the Krull dimension of \( \{ \mathcal{P} \} \) in \( \text{Spc}(\mathcal{K}) \), or the opposite of its Krull codimension. Assuming \( \text{dim}(\mathcal{P}) \) is clear from the context, we shall use the notation \( \text{Spc}(\mathcal{K})_{(p)} := \{ \mathcal{P} \in \text{Spc}(\mathcal{K}) \mid \text{dim}(\mathcal{P}) = p \} \).

Remark 3. In my opinion, there is nothing conceptually remarkable about the free abelian group on \( \text{Spc}(\mathcal{K})_{(p)} \). Therefore I propose another definition of \( p \)-dimensional cycles. This requires some preparation.

Definition 4. Recall from [3, § 4] that a rigid tensor triangulated category \( \mathcal{L} \) is called local if \( a \otimes b = 0 \) implies \( a = 0 \) or \( b = 0 \). Conceptually, this means that \( \text{Spc}(\mathcal{L}) \) is a local space, i.e. that \( \text{Spc}(\mathcal{L}) \) has a unique closed point \( \ast := 0 \subset \mathcal{L} \), which is prime by assumption.

Example 5. For every prime \( \mathcal{P} \in \text{Spc}(\mathcal{K}) \), the following tensor triangulated category is local in the above sense:

\[
\mathcal{K}_\mathcal{P} := \left( \mathcal{K}/\mathcal{P} \right)^\natural
\]

where \( \mathcal{K}/\mathcal{P} \) denotes the Verdier quotient and \((\cdot)^\natural\) the idempotent completion. We call \( \mathcal{K}_\mathcal{P} \) the local category at \( \mathcal{P} \). There is an obvious (localization) functor

\[
q_\mathcal{P} : \mathcal{K} \to \mathcal{K}/\mathcal{P} \to \mathcal{K}_\mathcal{P}
\]

composed of localization and idempotent completion. (The category \( \mathcal{K}_\mathcal{P} \) can also be understood as the strict filtered colimit of the \( \mathcal{K}(U) \) over those open subsets \( U \subset \text{Spc}(\mathcal{K}) \) which contain \( \mathcal{P} \). See more in [4, § 2.2] if helpful.) We can identify \( \text{Spc}(\mathcal{K}_\mathcal{P}) \) with the subspace \( \{ \mathcal{Q} \in \text{Spc}(\mathcal{K}) \mid \mathcal{P} \in \{ \mathcal{Q} \} \} \) of \( \text{Spc}(\mathcal{K}) \), hence the space \( \text{Spc}(\mathcal{K}_\mathcal{P}) \) remains noetherian.

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\( \mathcal{L} \).

As the previous comment suggests, if we take \( \mathcal{L} \) to have finite-length (in the categorical sense that they admit a finite filtration with simple subquotients). The present notation, \( \text{Min}(\mathcal{L}) \), is less biased towards commutative algebra and therefore probably preferable. It is however an interesting question to find some structure theorems about \( \text{Min}(\mathcal{L}) \).

Remark 7. Some comments are in order:

1. This subcategory was called the subcategory of finite-length objects in [2] and denoted \( \text{FL}(\mathcal{L}) \). As far as I know, there is no reason for objects of \( \text{Min}(\mathcal{L}) \) to have finite-length (in the categorical sense that they admit a finite filtration with simple subquotients). The present notation, \( \text{Min}(\mathcal{L}) \), is less biased towards commutative algebra and therefore probably preferable. It is however an interesting question to find some structure theorems about \( \text{Min}(\mathcal{L}) \).

2. As the previous comment suggests, if we take \( \mathcal{L} = \text{Kb}(\text{R-proj}) \) the category of perfect complexes for \( \text{R} \) noetherian and local, then \( \mathcal{L} \) is local and \( \text{Min}(\mathcal{L}) \) is the subcategory of perfect complexes with finite-length homology.

3. One can of course consider \( \text{Min}(\mathcal{L}) \) even if \( * \) is not Thomason but in that case it would just be the zero subcategory \( 0 = \mathcal{L}_\emptyset \).

Definition 6. Assuming that \( \mathcal{L} \) is local and that \( \text{Spc}(\mathcal{L}) \) is noetherian, the open complement of the unique closed point \( \{ * \} \) in \( \text{Spc}(\mathcal{L}) \) is quasi-compact, i.e. \( \{ * \} \) is a “Thomason (closed) subset”. Under the classification of thick \( \otimes \)-ideals of \( \mathcal{L} \), see [1], this one-point subset corresponds to the minimal non-zero thick \( \otimes \)-ideal

\[
\text{Min}(\mathcal{L}) := \mathcal{L}_{\{ * \}} = \{ a \in \mathcal{L} \mid \text{supp}(a) \subseteq \{ * \} \}.
\]

These are the objects with minimal possible support (empty or a point).

Definition 8. Let \( p \in \mathbb{Z} \). We define the group of generalized \( p \)-cycles to be

\[
\mathcal{Z}_p(\mathcal{K}) := \bigoplus_{\mathcal{P} \in \text{Spc}(\mathcal{K})} K_0\left( \text{Min}(\mathcal{K}_p) \right),
\]

where \( K_0 \) is the Grothendieck \( K \)-group (the quotient of the monoid of isomorphism classes \( [a] \) of objects under \( \oplus \), by the submonoid of those \( [a] + [\Sigma b] + [c] \) for which there exists a distinguished triangle \( a \rightarrow b \rightarrow c \rightarrow \Sigma a \).

Out of nostalgia for usual cycles, a generalized \( p \)-cycle can be written \( \sum_p \lambda_{\mathcal{P}} \cdot \mathcal{P} \) or \( \sum_p \lambda_{\mathcal{P}} \cdot \{ \mathcal{P} \} \), for \( \lambda_{\mathcal{P}} \in K_0\left( \text{Min}(\mathcal{K}_p) \right) \). This is a purely notational choice. The non-trivial point is that we allow coefficients \( \lambda_{\mathcal{P}} \) to live in other abelian groups than \( \mathbb{Z} \), namely the Grothendieck groups of the minimal categories at every \( \mathcal{P} \).

Example 9. Let \( X \) be a (topologically) noetherian scheme and \( \mathcal{K} = \text{D}^\text{perf}(X) \) the derived category of perfect complexes, whose spectrum \( \text{Spc}(\mathcal{K}) \cong X \) recovers the underlying space of \( X \). Let \( \text{dim}(\cdot) \) be the (opposite of the) Krull (co)dimension. Then we recover the usual \( p \)-dimensional (resp. \( -p \)-codimensional) cycles. Indeed, we have by Thomason that \( \mathcal{K}_p \cong \text{Kb}(\mathcal{O}_{X,x} - \text{proj}) \) if \( \mathcal{P} \in \text{Spc}(\mathcal{K}) \) corresponds to \( x \in X \). The reason why integral coefficients suffice over regular schemes is that the group homomorphism defined by alternate sum of length of homology groups

\[
K_0\left( \text{Min}(\text{Kb}(\mathcal{O}_{X,x} - \text{proj})) \right) \rightarrow \mathbb{Z},
\]

is an isomorphism if \( X \) is regular (at \( x \)). However, in general, the left-hand group could be tricky, as discussed for instance in Roberts-Srinivas [6].

Now to the relations. There might be several definitions of relations. The most flexible and most obvious one is the following.

Definition 10. For a (specialization) closed subset \( Y \subset \text{Spc}(\mathcal{K}) \), we set \( \text{dim}(Y) = \sup \{ \text{dim}(\mathcal{P}) \mid \mathcal{P} \in Y \} \) and consider the filtration \( \cdots \subset \mathcal{K}_p \subset \mathcal{K}_{(p+1)} \subset \cdots \subset \mathcal{K} \) by dimension of support

\[
\mathcal{K}_p := \{ a \in \mathcal{K} \mid \text{dim(supp}(a)) \leq p \}.
\]
By [2, Thm. 3.24], localization induces an equivalence

\[(\mathcal{K}(p)/\mathcal{K}(p-1))^\mathbb{Z} \cong \coprod_{\mathcal{Y} \in \text{Spec}(\mathcal{K})} \text{Min}(\mathcal{K}),\]

and consequently \(Z_p(\mathcal{K}) \cong K_0\left((\mathcal{K}(p)/\mathcal{K}(p-1))^\mathbb{Z}\right)\). Note that this definition of \(Z_p(\mathcal{K})\) does not need \(\text{Spec}(\mathcal{K})\) being noetherian. It also allows the definition of the \(p\)-boundaries \(B_p(\mathcal{K})\) as the image in \(Z_p(\mathcal{K})\) of \(\text{Ker}\left(K_0(\mathcal{K}(p)) \to K_0(\mathcal{K}(p+1))\right)\). In other words we have the diagram with exact rows

\[
\begin{array}{ccc}
\text{Ker}(i) & \longrightarrow & K_0(\mathcal{K}(p)) \quad \longrightarrow \quad K_0(\mathcal{K}(p+1)) \\
\downarrow & & \downarrow \\
B_p(\mathcal{K}) & \longrightarrow & Z_p(\mathcal{K}) \quad \longrightarrow \quad CH_p(\mathcal{K})
\end{array}
\]

in which we define \(CH_p(\mathcal{K}) := Z_p(\mathcal{K})/B_p(\mathcal{K})\) to be the quotient of \(p\)-cycles by \(p\)-boundaries. These groups could be called the \((K\text{-theoretic)}\) \(CH\) groups of \(p\)-cycles in \(\mathcal{K}\), with respect to the chosen dimension function \(\text{dim}\).

**Remark 12.** The above \(\text{Ker}(i)\) is an \textit{ad hoc} replacement for the maybe more natural image of \(K_1(\mathcal{K}(p+1)/\mathcal{K}(p))\) by a connecting homomorphism. The reason for the above definition is that triangulated categories do not behave well with higher \(K\)-theory. However, with this definition, it is not too hard to check that \(CH_p(\mathcal{K}) = CH_p(\mathcal{X})\) when \(\mathcal{X}\) is a regular scheme and \(\mathcal{X} = \text{D}^\text{perf}(\mathcal{X})\). See more in Klein [5].

It is however tempting to give another definition of \(p\)-boundaries, closer to the classical ideas of equivalence of \(p\)-cycles by means of divisors of rational functions on \((p + 1)\)-dimensional varieties. We need a preparation.

**Lemma 13.** Let \(a \in \mathcal{K}(p+1)\) be an object with support of dimension at most \(p + 1\) and let \(\gamma : a \sim a\) be an automorphism in \(\mathcal{K}(p+1)/\mathcal{K}(p)\). Choose a fraction \(a \xrightarrow{\alpha} b \xleftarrow{\beta} a\) in \(\mathcal{K}(p+1)\) representing \(\gamma\), so that \(\text{cone}(\alpha)\) and \(\text{cone}(\beta)\) both belong to \(\mathcal{K}(p)\). Then the difference \([\text{cone}(\alpha)] - [\text{cone}(\beta)]\) in \(K_0(\mathcal{K}(p))\) belongs to \(\text{Ker}\left(K_0(\mathcal{K}(p)) \to K_0(\mathcal{K}(p+1))\right)\) and is independent of the choice of \(\alpha\) and \(\beta\).

**Proof.** This is an immediate verification: In \(K_0(\mathcal{K}(p+1))\), we have \([\text{cone}(\alpha)] = [b] - [a] = [\text{cone}(\beta)]\), hence the first statement. Independence on the choice of the fraction up to amplification by a morphism \(s : b \to b'\) with cone in \(\mathcal{K}(p)\) follows by the octahedron axiom: \([\text{cone}(sa)] = [\text{cone}(s)] + [\text{cone}(\alpha)]\) and \([\text{cone}(s\beta)] = [\text{cone}(s)] + [\text{cone}(\beta)]\), so \([\text{cone}(sa)] - [\text{cone}(s\beta)] = [\text{cone}(\alpha)] - [\text{cone}(\beta)]\].

**Definition 14.** Let \(a \in \mathcal{K}(p+1)\) and let \(\gamma : a \sim a\) be an automorphism in \(\mathcal{K}(p+1)/\mathcal{K}(p)\). Choose a fraction \(a \xrightarrow{\alpha} b \xleftarrow{\beta} a\) in \(\mathcal{K}(p+1)\) representing \(\gamma\), and let

\[
\text{div}(a \sim a) = [q(\text{cone}(\alpha))] - [q(\text{cone}(\beta))] \in B_p(\mathcal{K})
\]

where \(q : \mathcal{K}(p) \to (\mathcal{K}(p)/\mathcal{K}(p-1))^\mathbb{Z}\) is the canonical functor. We might call this element the \textit{divisor} of \(\gamma : a \sim a\). This generalized \(p\)-cycle is a \(p\)-boundary by construction.

**Remark 15.** Of course, in view of the equivalence (11), we can also write

\[
\text{div}(\gamma) = \sum_{\mathcal{Y} \in \text{Spec}(\mathcal{K})} [q_{\mathcal{Y}}(\text{cone}(\alpha))] - [q_{\mathcal{Y}}(\text{cone}(\beta))]
\]
where \( q_p : \mathcal{K} \to \mathcal{K}_p \) is the localization and where \( \gamma = (a \xrightarrow{\alpha} b \xleftarrow{\beta} a) \) as before. The above formula for the divisor might look more familiar to the reader.

**Remark 16.** A priori, there might be more \( p \)-boundaries than the ones coming from the above divisors \( \text{div}(\gamma) \). This means that one might have a different Chow group \( \text{CH}'_p(\mathcal{K}) \) defined as the quotient of \( \mathbb{Z}_p(\mathcal{K}) \) by the subgroup generated by those \( \text{div}(\gamma) \). This group \( \text{CH}'_p(\mathcal{K}) \) would surject the group \( \text{CH}_p(\mathcal{K}) \) of Definition 10. However, in the case of \( \mathcal{K} = \text{D}^{\text{perf}}(X) \) for a (nice) regular scheme \( X \), it might well be that \( \text{CH}'_p \) coincides with \( \text{CH}_p \) because all relations coming from \( K_1 \) seem to be captured by divisors. This point requires further investigation and we refer the interested reader to the forthcoming [5].

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**References**


