

GLUING REPRESENTATIONS VIA IDEMPOTENT MODULES AND CONSTRUCTING ENDOTRIVIAL MODULES

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ABSTRACT. Let G be a finite group and k be a field of characteristic p . We show how to glue Rickard idempotent modules for a pair of open subsets of the cohomology variety along an automorphism for their intersection. The result is an endotrivial module. An interesting aspect of the construction is that we end up constructing finite dimensional endotrivial modules using infinite dimensional Rickard idempotent modules. We prove that this construction produces a subgroup of finite index in the group of endotrivial modules. More generally, we also show how to glue any pair of kG -modules.

CONTENTS

| | |
|--------------------------------------------------|----|
| Introduction | 1 |
| 1. Stable categories and support varieties | 3 |
| 2. Endotrivial modules | 5 |
| 3. U -isomorphisms and localisation | 6 |
| 4. Gluing finitely generated modules | 10 |
| 5. Rickard's idempotent modules | 12 |
| 6. Gluing arbitrary modules | 14 |
| 7. A gluing construction of endotrivial modules | 19 |
| 8. First example: Rank two | 22 |
| 9. Second example: The dihedral group D_8 | 23 |
| 10. The rank of the group of endotrivial modules | 24 |
| 11. Gluing along endomorphisms | 28 |
| References | 31 |

INTRODUCTION

Suppose that G is a finite group and k is a field of characteristic $p > 0$. The endotrivial kG -modules are the elements of the Picard group of invertible objects in the stable category of kG -modules. They form an important subgroup of the group of all self equivalences of the stable category. In addition, endotrivial modules play a significant role in the block

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theory and modular representation theory of G . There is now a complete classification of endotrivial modules in the case that the group G is a p -group; see [11, 12]. For the most part, the endotrivial modules over a p -group are all Heller shifts of the trivial module. In the cases where there are unusual endotrivial modules, various constructions for the exotic examples exist.

In this paper we investigate another construction of endotrivial modules, which is not limited to p -groups. The method uses infinite dimensional techniques, based on a construction of Balmer and Favi [3], to produce finite dimensional endotrivial modules. We analyse the new method by comparing it to one of the two constructions of Carlson [11]. We show that the “cohomological pushout” method of [11] is a “gluing” in the sense of [3].

Geometrically, the “gluing” construction can be interpreted as taking the patching data for an invertible sheaf over the variety associated to the cohomology ring $H^*(G, k)$ and translating these data into patching data for infinite dimensional Rickard idempotent modules. The results are finite dimensional endotrivial modules. The significance of the idempotent modules is that each one we use is naturally isomorphic to the trivial module in a particular localisation associated to the patching data of the stable category. Note that there are problems with trying to do the patching with more than two open sets, but fortunately using two open sets already produces enough modules to generate a subgroup of finite index in the group of endotrivial modules.

In Sections 1 and 2, we present background material, definitions and notation on stable categories and on endotrivial modules. One notable result is that an invertible object in the stable category of all (not just finitely generated) kG -modules must be the direct sum of a finitely generated endotrivial module and a projective module. In Section 3, we give the fundamentals of localisation and verify that the inclusion functor of a localised category of finitely generated modules is fully faithful into the corresponding localised category of all kG -modules. The gluing process is introduced and defined in Section 4. There it is shown that the “cohomological pushouts” of [11] are in fact gluings. In the two Sections that follow, we recall some facts about Rickard’s idempotent modules and show how they can be used to glue not necessarily finitely generated modules, via an appropriate weak pullback. In Section 7, we unfold this construction when the two glued modules are both the trivial module k , and we verify that the modules so constructed are endotrivial and coincide with the collection of modules which can be constructed by the pushout method of [11].

Sections 8 and 9 include some explicit calculations. The main question is what modules actually are constructed from a gluing or weak pullback along a chosen automorphism of an idempotent module. We answer the question first for groups of rank 2, and then illustrate the process with some specific automorphisms in the case of the Klein four group and the dihedral group of order eight. In particular, we emphasize that the explicit modules we construct in these examples are not new, although our construction sheds new light.

In Section 10, we prove directly that the endotrivial modules, constructed from idempotent modules using the gluing method, give rise to a subgroup of finite index in the group of endotrivial modules. The final section is devoted to a variation on the construction of Section 7, using endomorphisms instead of automorphisms. In an earlier version of the

paper this method was used for the proof of the main theorem of Section 10, and may still be of some interest even though it is no longer essential for the main results of the paper.

1. STABLE CATEGORIES AND SUPPORT VARIETIES

Throughout the paper we let G denote a finite group and k a field of characteristic $p > 0$. If M is a kG -module, let $\Omega(M)$ denote the kernel of a projective cover $P \rightarrow M$ of M . Likewise, let $\Omega^{-1}(M)$ be the cokernel of an injective hull $M \rightarrow Q$. Recall that a kG -module is projective if and only if it is injective. So for any $n \in \mathbb{Z}$, we write $\Omega^n(M)$ for the appropriate iteration of Ω or Ω^{-1} applied to M .

Recall that the *stable module category* $\mathbf{StMod}(kG)$ is the additive quotient of the category $\mathbf{Mod}(kG)$ of all (left) kG -modules by the subcategory of projective modules. Explicitly, the category $\mathbf{StMod}(kG)$ has as objects the kG -modules, not necessarily finitely generated, and as morphisms the groups:

$$\underline{\mathrm{Hom}}_{kG}(M, N) := \frac{\mathrm{Hom}_{kG}(M, N)}{\mathrm{PHom}_{kG}(M, N)}$$

where $\mathrm{PHom}_{kG}(M, N)$ is the subgroup of those kG -homomorphisms which factor through a projective module (which we can always choose to be the injective envelope of M or projective cover of N). The category $\mathbf{StMod}(kG)$ is triangulated with suspension functor $\Sigma = \Omega^{-1}$, the inverse of the Heller functor Ω . The category $\mathbf{StMod}(kG)$ carries a tensor structure given by $M \otimes N = M \otimes_k N$ with G acting diagonally.

We assume the reader has some minimal knowledge of triangulated categories in general, and at least of $\mathbf{StMod}(kG)$, as can be acquired in [5, 9, 23].

We denote by $\mathbf{stmod}(kG)$ the full subcategory of $\mathbf{StMod}(kG)$ on those M which are isomorphic in $\mathbf{StMod}(kG)$ to a finitely generated module. In fact, $\mathbf{stmod}(kG)$ is precisely the subcategory of compact objects of $\mathbf{StMod}(kG)$ (an object X is said to be *compact* if homomorphisms out of it distribute over direct sums, in the sense that the natural map $\bigoplus_{\alpha} \mathrm{Hom}(X, Y_{\alpha}) \rightarrow \mathrm{Hom}(X, \bigoplus_{\alpha} Y_{\alpha})$ is an isomorphism). This category $\mathbf{stmod}(kG)$ is equivalent to its more usual description as the additive quotient of the category $\mathbf{mod}(kG)$ of finitely generated kG -modules by the subcategory $\mathbf{proj}(kG)$ of projective objects.

The Krull-Schmidt Theorem holds for kG -modules. So, working in $\mathbf{StMod}(kG)$ simply consists in forgetting projective summands. The statement “ $M \cong N$ in $\mathbf{StMod}(kG)$ ” means $M \oplus (\mathbf{proj}) \cong N \oplus (\mathbf{proj})$ in the usual notation. We systematically drop the “ $\oplus (\mathbf{proj})$ ” in the sequel since this summand vanishes in our triangulated categories. For instance, for $M \in \mathbf{StMod}(kG)$, the statement “ M is finitely generated” should be understood as $M \in \mathbf{stmod}(kG)$, i.e., $M \cong M_0 \oplus (\mathbf{proj})$ with M_0 finitely generated.

The *projective support variety* of G over the field k is the projective variety (or rather the scheme)

$$\mathcal{V}_G = \mathrm{Proj} H^*(G, k).$$

In general, if R is a graded commutative k -algebra, we define $\mathrm{Proj} R$ to be the scheme whose underlying space is the set of homogeneous prime ideals of R not containing the maximal

ideal $\mathfrak{m} = R^+$ generated by elements of positive degree, with the Zariski topology. So if \mathfrak{a} is a homogeneous ideal in R then $\mathcal{V}(\mathfrak{a})$ is the closed set $\{\mathfrak{p} \in \mathbf{Proj} R \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

If $\mathfrak{p} \in \mathbf{Proj} R$ then we write $R_{\mathfrak{p}}$ for the homogeneous localisation whose elements of degree n are the quotients x/y with $y \notin \mathfrak{p}$ and $\deg x - \deg y = n$. The degree zero part of this localisation gives the stalk $\mathcal{O}_{X,\mathfrak{p}}$ of the structure sheaf \mathcal{O}_X of $X = \mathbf{Proj} R$ (see Hartshorne [16] §II.2 for further details).

Every module M has a support variety $\mathcal{V}_G(M) \subseteq \mathcal{V}_G$. When M is finitely generated, $\mathcal{V}_G(M)$ is the closed set $\mathcal{V}_G(M) = \mathcal{V}(\mathfrak{a})$ defined by the ideal $\mathfrak{a} = \text{Ann}_{H^*(G,k)}(\text{Ext}_{kG}^*(M, M))$, where $H^*(G, k) = \text{Ext}_{kG}^*(k, k)$ acts on $\text{Ext}_{kG}^*(M, M)$ in the natural way. (See for example [4], Chapter 5.) When M is not finitely generated, see [6].

The assignment $M \mapsto \mathcal{V}_G(M)$ satisfies a few easy rules (see [9]). In the language of [2], the support variety $\mathcal{V}_G \cong \text{Spc}(\mathbf{stmod}(kG))$ is the *spectrum* of the tensor triangulated category $\mathbf{stmod}(kG)$. This allows us to use the gluing technique of [3], which we briefly recall in Section 4.

We shall use Quillen's Dimension Theorem [19, 20], which says that

$$\mathcal{V}_G = \bigcup_E \text{res}_{G,E}^*(\mathcal{V}_E)$$

where the union is over a set of representatives of the maximal elementary abelian p -subgroups of G . In particular, any image $\text{res}_{G,E}^*(\mathcal{V}_E)$, for E a maximal elementary abelian p -subgroup, is a component of the variety $\mathbf{Proj} H^*(G, k)$. Moreover, we have that if $\zeta \in H^*(G, k)$ is an element whose restriction to every elementary abelian p -subgroup is zero, then ζ is nilpotent.

Example 1.1. We warn the reader that the generators of $H^*(G, k)$ are usually not all in the same degree, so that the usual intuitions from projective geometry might fail. Here is an example of this phenomenon.

Let $p \geq 5$ be a prime, let G be the semidirect product $(\mathbb{Z}/p)^3 \rtimes \Sigma_3$, with the permutation action, and let k be a field of characteristic p . Then

$$H^*(G, k) = k[x_2, x_4, x_6] \otimes \Lambda(y_1, y_3, y_5)$$

where the subscript indicates the degree of the generator and $\Lambda(\dots)$ denotes an exterior algebra. So $H^*(G, k)$ modulo its radical is a polynomial ring $k[x_2, x_4, x_6]$, and $\mathbf{Proj} H^*(G, k)$ is equal to $\mathbf{Proj} k[x_2, x_4, x_6]$ as a variety (though not as a scheme because of the nilpotent part).

We claim that the variety $\mathbf{Proj} k[x_2, x_4, x_6]$ is singular. To see this, observe that the coordinate ring for the affine patch $x_4 \neq 0$ is generated by $u = x_2^2/x_4$, $v = x_6^2/x_4^3$ and $w = x_2x_6/x_4^2$, which satisfy the relation $uv = w^2$. So there is a singularity at the point $u = v = w = 0$ of this patch, which is the projective point $(0 : 1 : 0)$.

Example 1.2. In the case of an elementary abelian p -group $G = (\mathbb{Z}/p)^r$, $\mathbf{Proj} H^*(G, k)$ is projective space $\mathbb{P}^{r-1}(k)$. In this case, Dade's Theorem [14] states that every endotrivial module is isomorphic to $\Omega^n k$ for some $n \in \mathbb{Z}$. This can be compared with the invertible

sheaves on projective space: every invertible sheaf on $\mathbb{P}^{r-1}(k)$ is isomorphic to $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$, see for instance Hartshorne [16, Corollary II.6.17].

2. ENDOTRIVIAL MODULES

Endotrivial modules and endopermutation modules were first named in [14] by Dade who showed that the sources (in the context of the theory of vertices and sources) of irreducible module for p -nilpotent groups are endopermutation. In addition, the classes of endotrivial modules modulo projective modules make up the Picard group of invertible modules in $\mathbf{stmod}(kG)$ as well as playing an important role in block theory. In this section we give some discussion of endotrivial modules and end with a proof that any invertible object in $\mathbf{StMod}(kG)$ is actually an endotrivial module in $\mathbf{stmod}(kG)$.

As originally defined, a finitely generated module is endotrivial if its k -endomorphism ring is isomorphic in $\mathbf{stmod}(kG)$ to the trivial module. That is, M is endotrivial if

$$\mathrm{Hom}_k(M, M) \cong k \oplus P$$

for some projective module P . Since for any finitely generated kG -modules M and N , $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$, it is equivalent to say that M is endotrivial if $M^* \otimes M$ is isomorphic to the trivial module in $\mathbf{stmod}(kG)$. A complete classification of the endotrivial modules for a p -group was finally completed in [11, 12]. The answer briefly is that the Picard group of endotrivial modules has no torsion unless G is cyclic or $p = 2$ and G is a quaternion or semi-dihedral group. Moreover, the torsion free part of the group is generated by $\Omega(k)$ unless G has at least two conjugacy classes of maximal elementary abelian subgroups and at least one of the classes has rank 2. This last fact which was first proved by Alperin [1] for p -groups, holds for any finite group.

Of relevance for this paper is a construction in [11]. We should emphasize that there are actually two methods for constructing endotrivial module given in that paper. The first method, which we might call the “sectional” method creates endotrivial module as sections U/V where U and V are very carefully chosen submodules

$$\{0\} \subseteq V \subseteq U \subseteq \Omega^n(k).$$

The choice of U and V is dictated by a somewhat complicated formula determined by the structure of the cohomology ring $H^*(G, k)$. Every endotrivial module for a p -group can be constructed using the sectional method.

Of interest in this paper is the second method, which we call the “cohomological-pushout” method. It requires finding an element in $H^*(G, k)$ which has nontrivial restriction to the center of a Sylow p -subgroup of G , and then taking a pushout along a homomorphism whose existence is guaranteed by the support variety of the corresponding Carlson module. The method is described in Section 4 (see Diagram (4.1)) of this paper, where we prove that the construction is a gluing. It is important to note that not every endotrivial module can be obtained by this method. Specifically, we do not get any of the torsion modules for the quaternion or semi-dihedral groups and there are a few elements of infinite order that we don’t get (see Example 8.2 of *loc. cit.*). On the other hand, the method is guaranteed to

produce generators for a subgroup of finite index of the group of endotrivial modules. And most importantly, it is applicable to all finite groups, not just p -groups.

We end the section with the promised result on the finite generation of endotrivial modules.

Theorem 2.1. *Suppose that a module $M \in \mathcal{T} = \mathbf{StMod}(kG)$ is invertible in the sense that there exists a module N with $M \otimes N \cong k$ in \mathcal{T} . Then M and N belong to $\mathbf{stmod}(kG)$ and $N \cong M^*$ in \mathcal{T} .*

Proof. The assumption $M \otimes N \cong k$ implies that $M \otimes - : \mathcal{T} \rightarrow \mathcal{T}$ is an equivalence of categories, with inverse $N \otimes -$. So, for any object Y in \mathcal{T} , we have an isomorphism, natural in Y :

$$\mathrm{Hom}_{\mathcal{T}}(k, N \otimes Y) \cong \mathrm{Hom}_{\mathcal{T}}(M, M \otimes N \otimes Y) \cong \mathrm{Hom}_{\mathcal{T}}(M, Y)$$

using successively the fact that $M \otimes -$ is an equivalence and the assumption $M \otimes N \cong k$. Naturality implies in particular that if $Y \hookrightarrow M$ is a submodule, the following diagram commutes (ignore 1_M , f , g and h which appear later in the proof):

$$\begin{array}{ccc} g \in \mathrm{Hom}_{\mathcal{T}}(k, N \otimes Y) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{T}}(M, Y) \ni h \\ \downarrow & & \downarrow \\ f \in \mathrm{Hom}_{\mathcal{T}}(k, N \otimes M) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{T}}(M, M) \ni 1_M. \end{array} \quad (2.1)$$

Now take the identity $1_M \in \mathrm{Hom}_{\mathcal{T}}(M, M)$. There is a morphism $f: k \rightarrow N \otimes M$ which maps to 1_M . Since k is finitely generated, we can find a finitely generated submodule $Y \subseteq M$ such that f factors as $k \xrightarrow{g} N \otimes Y \rightarrow N \otimes M$. Pushing this element $g \in \mathrm{Hom}_{\mathcal{T}}(k, N \otimes Y)$ into $\mathrm{Hom}_{\mathcal{T}}(M, Y)$, we find a morphism $h: M \rightarrow Y$ which maps to 1_M , i.e., h is a section of the inclusion $Y \hookrightarrow M$ in \mathcal{T} . This means that M is a direct summand of $Y \in \mathbf{stmod}(kG)$ and so $M \in \mathbf{stmod}(kG)$.

The isomorphism $N \cong M^*$ is well known: the category $\mathbf{stmod}(kG)$ is closed symmetric monoidal and so any invertible object has its dual as inverse. See if necessary [17, Proposition A.2.8]. \square

Remark 2.2. Alternatively to the above direct proof, one can use [17, Proposition A.2.8] at the cost of checking that $\mathbf{StMod}(kG)$ satisfies the hypotheses of *loc. cit.* and that $\mathbf{stmod}(kG)$ consists exactly of the compact objects of $\mathbf{StMod}(kG)$. These facts are of independent interest.

3. U -ISOMORPHISMS AND LOCALISATION

In this section we establish a few basic facts and notations concerned with localisations. In particular, we show that the functor on the localised subcategories, induced by the inclusion $\mathbf{stmod} \hookrightarrow \mathbf{StMod}$ is fully faithful (see 3.8).

Definition 3.1. Let $W \subseteq \mathcal{V}_G$ be a closed subset. Consider the following subcategories of $\mathbf{StMod}(kG)$:

$$\begin{array}{ccc} \mathcal{C}_W & \subseteq & \mathcal{C}_W^\oplus \\ \cap & & \cap \\ \mathbf{stmod}(kG) & \subseteq & \mathbf{StMod}(kG), \end{array}$$

where \mathcal{C}_W is the full subcategory of $\mathbf{stmod}(kG)$ consisting of those objects whose support is contained in W . Note that \mathcal{C}_W is a \otimes -ideal thick subcategory of $\mathbf{stmod}(kG)$. The category \mathcal{C}_W^\oplus is the subcategory of $\mathbf{StMod}(kG)$ generated by \mathcal{C}_W in any of the following equivalent senses:

- (1) \mathcal{C}_W^\oplus is the smallest triangulated subcategory of $\mathbf{StMod}(kG)$ containing \mathcal{C}_W and closed under arbitrary coproducts (hence the notation, although \mathcal{C}_W^{\coprod} would be more precise).
- (2) \mathcal{C}_W^\oplus is the subcategory of $\mathbf{StMod}(kG)$ of those objects which are filtered colimits in $\mathbf{Mod}(kG)$ of objects of \mathcal{C}_W . (In the notation of [21, §5, Theorem 5.17], $\mathcal{C}_W^\oplus = \mathcal{C}_W^\rightarrow$.)
- (3) \mathcal{C}_W^\oplus is the subcategory of $\mathbf{StMod}(kG)$ consisting of those modules M which have the property that any morphism $L \rightarrow M$ with L finitely generated, factors through some object of \mathcal{C}_W . (This is equivalent to the above by [5, Theorem 5.3].)

It follows from [21, Prop. 5.9] that \mathcal{C}_W^\oplus is a \otimes -ideal in $\mathbf{StMod}(kG)$.

Definition 3.2. Given a morphism s in $\mathbf{StMod}(kG)$ and an open subset $U \subseteq \mathcal{V}_G$, we shall say that s is a U -isomorphism if the cone of s belongs to \mathcal{C}_W^\oplus where $W = \mathcal{V}_G \setminus U$ is the closed complement of U . For example, the morphism $\varepsilon_W: k \rightarrow F(W)$ of Theorem 5.1 below is a (prototypical) U -isomorphism. Also note that since \mathcal{C}_W^\oplus is a \otimes -ideal, we have that for every U -isomorphism $s: M \rightarrow N$ and for every object $L \in \mathbf{StMod}(kG)$, the morphism $L \otimes s: L \otimes M \rightarrow L \otimes N$ is again a U -isomorphism, for its cone is $L \otimes \text{cone}(s) \in \mathcal{C}_W^\oplus$.

The following is an easy application of the octahedral axiom (see [3, Lemma 1.13]):

Proposition 3.3. Consider, in a triangulated category, a distinguished triangle as follows:

$$X_1 \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} X_2 \oplus X_3 \xrightarrow{\begin{pmatrix} h & j \end{pmatrix}} X_4 \xrightarrow{\ell} \Sigma X_1.$$

Then $\text{cone}(f) \cong \text{cone}(j)$ and $\text{cone}(g) \cong \text{cone}(h)$. In particular, in the case of $\mathbf{StMod}(kG)$ and of $U \subseteq \mathcal{V}_G$ open, the morphism f is a U -isomorphism if and only if j is.

Definition 3.4. It is useful to say that a square

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ -g \downarrow & & \downarrow h \\ X_3 & \xrightarrow{j} & X_4 \end{array}$$

is a *weak pullback* if there exists a distinguished triangle as in the above statement, for some morphism $\ell: X_4 \rightarrow \Sigma X_1$, which will always remain of little relevance in the sequel. It is easy to check that X_1 has the property of the pullback of X_2 and X_3 above X_4 , except

for the uniqueness of the corner morphism (hence the “weak”). Similarly, this square is automatically a weak push-out.

We want to invert the U -isomorphisms, both in $\mathbf{stmod}(kG)$ and in $\mathbf{StMod}(kG)$. Roughly speaking, one can understand the resulting categories as the parts of $\mathbf{stmod}(kG)$ and of $\mathbf{StMod}(kG)$ which “lie over U ”, or equivalently, which survive after “killing” all objects which are supported on the closed complement of U . For the convenience of the reader, we recall some standard facts about localisations of triangulated categories.

Definition 3.5. We say that

$$\mathcal{J} \xrightarrow{j} \mathcal{K} \xrightarrow{q} \mathcal{L}$$

is an exact sequence of triangulated categories if $\mathcal{J} \subseteq \mathcal{K}$ is a thick subcategory of \mathcal{K} (i.e., for $X, Y \in \mathcal{K}$, $X \oplus Y \in \mathcal{J}$ implies $X \in \mathcal{J}$) and if

$$\mathcal{L} = \mathcal{K}/\mathcal{J}$$

is the *quotient* of \mathcal{K} by \mathcal{J} . The latter means that the functor q is the universal functor out of \mathcal{K} which sends \mathcal{J} to zero: $q \circ j = 0$. Equivalently, \mathcal{L} can be constructed as a *Verdier localisation*, see [22], as follows. Consider S the class of those morphisms $s: X \rightarrow Y$ in \mathcal{K} whose cone belongs to \mathcal{J} . Then $\mathcal{L} = S^{-1}\mathcal{K}$ is the localisation of \mathcal{K} with respect to the morphisms of S , i.e., the target of the universal functor out of \mathcal{K} which maps morphisms of S to isomorphisms.

Explicitly, we have a calculus of fractions. By this we mean that $\mathcal{L} = S^{-1}\mathcal{K}$ can be constructed as having the same objects as \mathcal{K} and morphisms between two objects X and Y being equivalence classes of left fractions $X \xleftarrow{s} Z \xrightarrow{f} Y$ where $s \in S$. Two fractions are declared equivalent if they have a common amplification $X \xleftarrow{s s'} Z' \xrightarrow{f s'} Y$ for $s': Z' \rightarrow Z$ in S . Equivalently, we can work with classes of right fractions $X \xrightarrow{g} W \xleftarrow{t} Y$ with $t \in S$. (The passage from left to right fractions is made by means of weak pushouts and weak pullbacks; see Proposition 3.3.)

The functor $q: \mathcal{K} \rightarrow S^{-1}\mathcal{K}$ is the identity on objects and sends a morphism $f: X \rightarrow Y$ to the class of the fraction $X \xleftarrow{1} X \xrightarrow{f} Y$. A morphism $f: X \rightarrow Y$ becomes zero in $S^{-1}\mathcal{K}$ if and only if there exists an $s: Z \rightarrow X$ in S such that $f s = 0$ or equivalently if there exists $t \in S$ such that $t f = 0$. A morphism $f: X \rightarrow Y$ becomes an isomorphism in $S^{-1}\mathcal{K}$ if and only if its cone belongs to \mathcal{J} . The subcategory \mathcal{J} is exactly the kernel of q .

Notation 3.6. Let us give short names to the various subcategories and Verdier localisations of the stable category which will appear below. We abbreviate:

$$\mathcal{T} := \mathbf{StMod}(kG) \quad \text{and} \quad \mathcal{C} := \mathbf{stmod}(kG)$$

and for any closed $W \subseteq \mathcal{V}_G$ with open complement $U = \mathcal{V}_G \setminus W$:

$$\begin{aligned} \mathcal{T}_W &:= \mathcal{C}_W^\oplus && \text{(see Definition 3.1)} \\ \mathcal{C}(U) &:= \mathcal{C}/\mathcal{C}_W = \mathbf{stmod}(kG)/\mathcal{C}_W \\ \mathcal{T}(U) &:= \mathcal{T}/\mathcal{T}_W = \mathbf{StMod}(kG)/\mathcal{C}_W^\oplus. \end{aligned}$$

In other words, $\mathcal{C}(U)$ and $\mathcal{T}(U)$ are the Verdier localisations with respect to U -isomorphisms of \mathcal{C} and \mathcal{T} respectively. We have the following commutative diagram of functors:

$$\begin{array}{ccccc} \mathcal{C}_W & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T}_W & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}(U). \end{array}$$

The rows are exact sequences of triangulated categories (Definition 3.5) and the left-hand first two vertical functors are the (fully faithful) inclusions of subcategories. The right-hand functor is the induced functor, which is fully faithful, as we now check.

Lemma 3.7. *Consider a full inclusion of triangulated categories $\mathcal{K} \subseteq \bar{\mathcal{K}}$ and classes of morphisms $S \subseteq \mathcal{K}$ and $\bar{S} \subseteq \bar{\mathcal{K}}$ such that $S \subseteq \bar{S}$. The induced functor $S^{-1}\mathcal{K} \rightarrow \bar{S}^{-1}\bar{\mathcal{K}}$ is fully faithful if the following condition holds: for any morphism $s: \bar{Z} \rightarrow X$ in \bar{S} with $X \in \mathcal{K}$, there exists a morphism $t: Z \rightarrow \bar{Z}$ with $Z \in \mathcal{K}$ and $st \in S$.*

Proof. This is an easy exercise on calculus of fractions. The condition implies that any fraction $X \xleftarrow{s} \bar{Z} \xrightarrow{f} Y$ with $X, Y \in \mathcal{K}$ but with $\bar{Z} \in \bar{\mathcal{K}}$ can be amplified into a fraction $X \xleftarrow{st} Z \xrightarrow{ft} Y$ with $Z \in \mathcal{K}$. This proves $S^{-1}\mathcal{K} \rightarrow \bar{S}^{-1}\bar{\mathcal{K}}$ full. The condition also implies that if $f \circ s = 0$ for some morphism $f: X \rightarrow Y$ in \mathcal{K} and for some morphism $s \in \bar{S}$, one has $f \circ (st) = 0$ with this time $st \in S$. This proves $S^{-1}\mathcal{K} \rightarrow \bar{S}^{-1}\bar{\mathcal{K}}$ faithful. \square

Proposition 3.8. *Let $W \subseteq \mathcal{V}_G$ be a closed subset with open complement U . The canonical functor $\mathcal{C}(U) \rightarrow \mathcal{T}(U)$ is fully faithful.*

Proof. Let us check the condition of Lemma 3.7 for $\mathcal{K} = \mathcal{C}$, $\bar{\mathcal{K}} = \mathcal{T}$ and S and \bar{S} the respective classes of U -isomorphisms. Let $s: \bar{Z} \rightarrow X$ be a U -isomorphism, i.e., a morphism whose cone belongs to \mathcal{T}_W , and assume that $X \in \mathcal{C}$, i.e., X is finitely generated. Consider a distinguished triangle:

$$\bar{Z} \xrightarrow{s} X \xrightarrow{s_1} \bar{Y} \xrightarrow{s_2} \Sigma \bar{Z}.$$

The object $\bar{Y} := \text{cone}(s)$ belongs to \mathcal{T}_W by hypothesis. By Property (3) of $\mathcal{T}_W = \mathcal{C}_W^\oplus$ in Definition 3.1, the morphism s_1 factors via some object $Y \in \mathcal{C}_W$, say $s_1 = uv: X \xrightarrow{v} Y \xrightarrow{u} \bar{Y}$. Let us write this in the middle square of the following diagram:

$$\begin{array}{ccccccc} Z & \xrightarrow{v_0} & X & \xrightarrow{v} & Y & \xrightarrow{v_2} & \Sigma Z \\ \downarrow \exists t & & \parallel & & \downarrow u & & \downarrow \Sigma t \\ \bar{Z} & \xrightarrow{s} & X & \xrightarrow{s_1} & \bar{Y} & \xrightarrow{s_2} & \Sigma \bar{Z}, \end{array}$$

where we also complete v into a distinguished triangle (first row). Let $t: Z \rightarrow \bar{Z}$ be a fill-in map as above. Of course, Z is finitely generated since Y and X are, and we have $\text{cone}(st) = \text{cone}(v_0) = Y \in \mathcal{C}_W$, i.e., the map st is a U -isomorphism as desired. \square

4. GLUING FINITELY GENERATED MODULES

In this section we define the gluing process and show that the cohomological-pushout method of [11, §4.5] is a gluing. For the entire section, assume that we have an open covering

$$\mathcal{V}_G = U_1 \cup U_2$$

of the projective support variety. We contemplate the following commutative diagram of localisations of triangulated categories (see Notations 3.6):

$$\begin{array}{ccccc} M \in \mathcal{C} = \mathbf{stmod}(kG) & \longrightarrow & \mathcal{C}(U_1) & \ni & M_1 \\ & & \downarrow & & \downarrow \\ M_2 \in \mathcal{C}(U_2) & \longrightarrow & \mathcal{C}(U_1 \cap U_2) & \cong & M_1 \cong M_2 \end{array}$$

The objects M , M_1 , M_2 are the ones of the following definition.

Definition 4.1. Consider two objects $M_1, M_2 \in \mathbf{stmod}(kG)$ for $i = 1, 2$ (though we think $M_i \in \mathcal{C}(U_i)$ as above). Suppose that we have an isomorphism $\sigma: M_1 \xrightarrow{\cong} M_2$ in $\mathcal{C}(U_1 \cap U_2)$. A *gluing of M_1 and M_2 along the isomorphism σ* is an object of $\mathbf{stmod}(kG)$ which is locally isomorphic to M_1 and M_2 in a compatible way with σ . That is, a gluing along σ is a triple (M, σ_1, σ_2) where M is an object of \mathcal{C} and

$$\sigma_1: M \xrightarrow{\cong} M_1 \quad \text{and} \quad \sigma_2: M \xrightarrow{\cong} M_2$$

are two isomorphisms in $\mathcal{C}(U_1)$ and $\mathcal{C}(U_2)$ respectively, such that the following diagram

$$\begin{array}{ccc} & & M_1 \\ & \nearrow^{\sigma_1} & \downarrow \sigma \\ M & & \\ & \searrow_{\sigma_2} & M_2 \end{array}$$

commutes in $\mathcal{C}(U_1 \cap U_2)$. Such a gluing always exists in \mathcal{C} and is unique up to isomorphism by [3, Corollary 5.10].

Applying this to $M_1 = k$ and $M_2 = k$, we obtain [3, Theorem 6.7], which is as follows.

Theorem 4.2 (Balmer-Favi). *Consider $\mathrm{Pic}(\mathcal{C}) = T_k(G)$ the group of finitely generated endotrivial kG -modules, i.e., the group of invertible objects in $\mathcal{C} = \mathbf{stmod}(kG)$ with respect to \otimes . Denote by $\mathbb{G}_m(U_1 \cap U_2) := \mathrm{Aut}_{\mathcal{C}(U_1 \cap U_2)}(k)$ the group of automorphisms of k in $\mathcal{C}(U_1 \cap U_2)$. Gluing two copies of k along an automorphism $\alpha \in \mathbb{G}_m(U_1 \cap U_2)$ defines a group homomorphism $\delta: \mathbb{G}_m(U_1 \cap U_2) \rightarrow \mathrm{Pic}(\mathcal{C})$.*

Remark 4.3. Observe that we can glue all sorts of objects M_1 and M_2 , not necessarily copies of k , and not only endotrivial modules. Even for the construction of endotrivial modules, it can be interesting to glue $\Omega^\ell k$ and $\Omega^m k$ along an isomorphism over $U_1 \cap U_2$. We illustrate this situation below.

We end the Section, with a proof that the construction [11, §4.5] is indeed a gluing construction. For the setting, suppose that G has at least two conjugacy classes of maximal elementary abelian p -subgroups and that at least one of these is a class of maximal elementary abelian subgroups of rank 2 (order p^2). Then the center Z of a Sylow p -subgroup of G must be cyclic. Choose a homogeneous element $\zeta \in H^m(G, k)$ with the property that ζ restricts to a non-nilpotent element of $H^*(Z, k)$. Consider a morphism $\zeta: \Omega^m k \rightarrow k$ in $\mathbf{stmod}(kG)$, representing the cohomology element ζ . Recall that the *Carlson module* L_ζ is defined by completing to a triangle $L_\zeta \rightarrow \Omega^m k \xrightarrow{\zeta} k$ in $\mathbf{stmod}(kG)$, and that $\mathcal{V}_G(L_\zeta)$ is the closed set $\mathcal{V}(\zeta)$ determined by the ideal (ζ) . See [9] for further details.

With the assumptions on G and ζ (and only with these assumptions), we know that $\mathcal{V}(\zeta) = W_1 \cup W_2$ decomposes into two disjoint non-empty closed subsets $W_1 \cap W_2 = \emptyset$, which is the only property we need for this construction. By [8] there is an analogous decomposition of the module $L_\zeta \cong L_1 \oplus L_2$ with the support of L_i being W_i for $i = 1, 2$. In other words, we have an exact sequence:

$$0 \longrightarrow L_1 \oplus L_2 \xrightarrow{(v_1 \ v_2)} \Omega^m k \xrightarrow{\zeta} k \longrightarrow 0.$$

Theorem 4.5 of [11] says that the module N obtained by the following push-out (marked \ulcorner):

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & L_2 & \xlongequal{\quad} & L_2 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & v_2 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L_\zeta & \xrightarrow{(v_1 \ v_2)} & \Omega^m k & \xrightarrow{\zeta} & k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 & & \begin{pmatrix} 1 & 0 \end{pmatrix} & & \ulcorner & & \rho \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_1 & \xrightarrow{\rho v_1} & N & \xrightarrow{\sigma_1} & k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{4.1}$$

is endotrivial. In triangular terms, $N \cong \text{cone}(v_2)$. Of particular interest are the morphisms σ_1 and ρ which appear in the above diagram and which satisfy $\sigma_1 \rho = \zeta$.

Theorem 4.4. *Consider the open complements $U_i = \mathcal{V}_G \setminus W_i$ of the above closed subsets W_i , for $i = 1, 2$. We have by assumption an open covering $\mathcal{V}_G = U_1 \cup U_2$. With the above notations, the module N is the gluing (Definition 4.1) of k and $\Omega^m k$ along the isomorphism $\zeta^{-1}: k \xrightarrow{\cong} \Omega^m k$ in $\mathcal{C}(U_1 \cap U_2)$.*

Proof. The exact sequences of diagram (4.1) yield corresponding triangles in $\mathbf{stmod}(kG)$. Since $\mathcal{V}_G(L_\zeta) = W_1 \cup W_2$, the morphism $\zeta: \Omega^m k \rightarrow k$ is a $U_1 \cap U_2$ -isomorphism (Definition 3.2), i.e., ζ is an isomorphism in $\mathcal{C}(U_1 \cap U_2)$. So, the statement makes sense. Since

$\mathcal{V}_G(L_i) = W_i$ for $i = 1, 2$, we see that $\sigma_1: N \rightarrow k$ is a U_1 -isomorphism and that $\rho: \Omega^m k \rightarrow N$ is a U_2 -isomorphism. Let us define $\sigma_2 := \rho^{-1}: N \xrightarrow{\cong} \Omega^m(k)$ in $\mathcal{C}(U_2)$. These are the desired isomorphisms $\sigma_1: N \xrightarrow{\cong} k$ in $\mathcal{C}(U_1)$ and $\sigma_2: N \xrightarrow{\cong} \Omega^m(k)$ in $\mathcal{C}(U_2)$. By (4.1), we have $\sigma_1 \rho = \zeta$. In $\mathcal{C}(U_1 \cap U_2)$, these three morphisms are isomorphisms by the above comments and hence the relation $\sigma_1 \rho = \zeta$ yields $\zeta^{-1} \sigma_1 = \rho^{-1} = \sigma_2$:

$$\begin{array}{ccc} & & k \\ & \nearrow^{\sigma_1} & \downarrow^{\zeta^{-1}} \\ N & & \Omega^m(k). \\ & \searrow_{\sigma_2} & \end{array}$$

This shows that N is the gluing of k and $\Omega^m(k)$ along ζ^{-1} , as in Definition 4.1. \square

5. RICKARD'S IDEMPOTENT MODULES

In this section, we recall some basic facts about *idempotent modules*. The definition, which is really an existence theorem is a part of the following.

Theorem 5.1 (Rickard [21]). *Suppose that W is a closed subset of \mathcal{V}_G .*

- (1) *There exist two kG -modules $E(W)$ and $F(W)$ and a distinguished triangle*

$$E(W) \xrightarrow{\eta_W} k \xrightarrow{\varepsilon_W} F(W) \xrightarrow{\theta_W} \Sigma E(W)$$

in $\mathbf{StMod}(kG)$ such that $E(W) \in \mathcal{C}_W^\oplus$ and such that $F(W)$ is \mathcal{C}_W^\oplus -local, i.e.,

$$\mathrm{Hom}(X, F(W)) = 0$$

for any $X \in \mathcal{C}_W^\oplus$.

- (2) *For any object $M \in \mathbf{StMod}(kG)$, the morphism $M \otimes \eta_W: M \otimes E(W) \rightarrow M \otimes k \cong M$ is the universal morphism from an object of \mathcal{C}_W^\oplus to M , and dually $M \otimes \varepsilon_W$ is the universal morphism from M to a \mathcal{C}_W^\oplus -local object. (In particular, $M \otimes F(W)$ is \mathcal{C}_W^\oplus -local.)*
- (3) *If $W_1, W_2 \subseteq \mathcal{V}_G$ are closed subsets, then there are unique isomorphisms*

$$E(W_1 \cap W_2) \cong E(W_1) \otimes E(W_2) \quad \text{and} \quad F(W_1 \cup W_2) \cong F(W_1) \otimes F(W_2)$$

such that $\eta_{W_1 \cap W_2} = \eta_{W_1} \otimes \eta_{W_2}$ and such that $\varepsilon_{W_1 \cup W_2} = \varepsilon_{W_1} \otimes \varepsilon_{W_2}$. In particular $E(W) \otimes E(W) \cong E(W)$ and $F(W) \otimes F(W) \cong F(W)$ (hence the name “idempotent” modules). Moreover, there exists two Mayer–Vietoris triangles

$$E(W_1 \cap W_2) \longrightarrow E(W_1) \oplus E(W_2) \longrightarrow E(W_1 \cup W_2) \longrightarrow \Sigma E(W_1 \cap W_2) \quad (5.1)$$

$$F(W_1 \cap W_2) \longrightarrow F(W_1) \oplus F(W_2) \longrightarrow F(W_1 \cup W_2) \longrightarrow \Sigma F(W_1 \cap W_2) \quad (5.2)$$

The proof is Rickard’s beautiful insight. The picky reader might observe that the key Lemma 4.2 in [21], is not correctly proved, and might even be incorrect as stated. With the same notation, the conclusion of that Lemma should be corrected to read: *Then there is a distinguished triangle*

$$X' \rightarrow \mathrm{hocolim} Y_i \rightarrow \mathrm{hocolim} Z_i \rightarrow \Sigma(X')$$

for some object X' which fits into a distinguished triangle $\bigoplus_i X_i \rightarrow \bigoplus_i X_i \rightarrow X' \rightarrow \bigoplus_i \Sigma X_i$. The problem is that X' might not be $\text{hocolim } X_i$. The proof of this statement is exactly the one given by Rickard in *loc. cit.* (Note that Verdier's result does not control the third morphism.) This Lemma is the starting point of the construction of idempotent modules and the reader can check that the above formulation suffices to prove the other statements of [21]. In particular, in the proof of Proposition 5.4, *loc. cit.*, $\mathcal{E}_{\mathcal{C}}(X)$ might not be $\text{hocolim}(A_i)$ but still belongs to \mathcal{C}^{\oplus} . The proof of Proposition 5.5, *loc. cit.*, remains the same.

Rickard's idempotent modules are extremely useful in that they allow the description of morphisms in the localisation $\mathcal{T}(U)$ in terms of usual morphisms in the stable category \mathcal{T} .

Proposition 5.2. *Let $W \subseteq \mathcal{V}_G$ be a closed subset with open complement U . Let $M, N \in \mathcal{T} = \text{StMod}(kG)$. We have an isomorphism*

$$\text{Hom}_{\mathcal{T}}(M, N \otimes F(W)) \xrightarrow{\cong} \text{Hom}_{\mathcal{T}(U)}(M, N \otimes F(W))$$

given by the localisation functor $\mathcal{T} \rightarrow \mathcal{T}(U)$.

Proof. This is a general fact coming from the situation guaranteed by Rickard's Theorem 5.1. In fact, the functor $- \otimes F(W)$ maps \mathcal{T} to the subcategory of \mathcal{T}_W -local objects and is a left adjoint to the inclusion of \mathcal{T}_W -local objects in \mathcal{T} . Then, the functor from \mathcal{T}_W -local objects to $\mathcal{T}/\mathcal{T}_W$ is an equivalence, or equivalently, one can realise the localisation functor as the functor $- \otimes F(W)$. Let us translate this in down-to-earth terms.

The key property of \mathcal{T}_W -local objects like $F(W)$ or $N \otimes F(W)$ is the following: Let $F \in \mathcal{T}$ be a \mathcal{T}_W -local object and let $t: F \rightarrow X$ be a U -isomorphism from our F to some object X . Then t is a split monomorphism. This is immediate from a distinguished triangle over t :

$$\begin{array}{ccccc} F & \xrightarrow{t} & X & \xrightarrow{t_1} & \text{cone}(t) \xrightarrow{t_2=0} \Sigma F \\ & \searrow \scriptstyle \exists r & \nearrow & & \end{array}$$

By definition of F being \mathcal{T}_W -local, the morphism t_2 must be zero because $\text{cone}(t) \in \mathcal{T}_W$. So, there exists $r: X \rightarrow F$ such that $rt = 1$.

With this in hand, the isomorphism of the statement is easy to prove. Let $F = N \otimes F(W)$. Any right fraction $M \xrightarrow{g} X \xleftarrow{t} F$ can be amplified by a retraction $r: X \rightarrow F$ as above, giving the equivalent fraction $M \xrightarrow{rg} F \xleftarrow{1} F$, that is a morphism coming from \mathcal{T} . Injectivity is proved similarly: if $tf = 0$ then $f = rtf = 0$. \square

Corollary 5.3. *Let $W \subseteq \mathcal{V}_G$ be closed with open complement U and let $M, N \in \mathcal{T}$. Then*

$$\text{Hom}_{\mathcal{T}}(M \otimes F(W), N \otimes F(W)) \cong \text{Hom}_{\mathcal{T}(U)}(M, N).$$

This isomorphism maps $\alpha: M \otimes F(W) \rightarrow N \otimes F(W)$ to $(N \otimes \varepsilon_W)^{-1} \circ \alpha \circ (M \otimes \varepsilon_W): M \rightarrow N$ in $\mathcal{T}(U)$. In particular, this isomorphism respects the composition operation and therefore, if we denote by $\text{Isom} \subseteq \text{Hom}$ the subsets of isomorphisms, we have an induced bijection

$$\text{Isom}_{\mathcal{T}}(M \otimes F(W), N \otimes F(W)) \cong \text{Isom}_{\mathcal{T}(U)}(M, N).$$

Proof. From Proposition 5.2, we know that localisation yields an isomorphism between $\mathrm{Hom}_{\mathcal{T}}(M \otimes F(W), N \otimes F(W))$ and $\mathrm{Hom}_{\mathcal{T}(U)}(M \otimes F(W), N \otimes F(W))$. In the latter group, we can replace $M \otimes F(W)$ by M and $N \otimes F(W)$ by N , since they are isomorphic in $\mathcal{T}(U)$ via $M \otimes \varepsilon_W$ and $N \otimes \varepsilon_W$ respectively. The isomorphism is exactly as announced in the statement. Hence it preserves composition. Therefore invertible elements, *i.e.*, isomorphisms, are also preserved. \square

Corollary 5.4. *Let $W \subseteq \mathcal{V}_G$ be a closed subset with open complement U . Then*

$$\mathrm{End}_{\mathcal{T}}(F(W)) \cong \mathrm{End}_{\mathcal{C}(U)}(k) \quad \text{and} \quad \mathrm{Aut}_{\mathcal{T}}(F(W)) \cong \mathrm{Aut}_{\mathcal{C}(U)}(k)$$

and these bijections send $\alpha: F(W) \rightarrow F(W)$ onto $\varepsilon_W^{-1} \alpha \varepsilon_W: k \rightarrow k$ in $\mathcal{C}(U)$.

Proof. Apply Corollary 5.3 to $M_1 = M_2 = k \in \mathcal{C}$ and replace $\mathrm{Hom}_{\mathcal{T}(U)}$ by $\mathrm{Hom}_{\mathcal{C}(U)}$ using the fact that $\mathcal{C}(U) \hookrightarrow \mathcal{T}(U)$ is fully faithful by Proposition 3.8. \square

Remark 5.5. It is known (see for example Benson and Gnacadja [7, §5.2]) that the endomorphism rings of Rickard idempotent modules are graded commutative.

Here is a useful example of the endomorphism ring of an idempotent module corresponding to a principal closed subset of \mathcal{V}_G .

Proposition 5.6. *Let $\zeta \in H^d(G, k)$ be a homogeneous element and consider the closed subset $W = \mathcal{V}_G(\zeta) \subseteq \mathcal{V}_G$ with open complement U . Then $\mathrm{End}_{\mathcal{T}}(F(W))$ is isomorphic to $(H^*(G, k)[\zeta^{-1}])^0$, the degree zero part of the cohomology ring localised at ζ . Via Corollary 5.4, a fraction $\frac{\eta}{\zeta^i}$ for $\eta \in H^{di}(G, k)$ corresponds to the fraction $k \xleftarrow{\zeta^i} \Omega^{di} k \xrightarrow{\eta} k$ in $\mathcal{C}(U)$.*

Proof. See Rickard [21, §6] or Friedlander and Pevtsova [15, Prop. 7.4]. \square

Remark 5.7. Let $U' \subseteq U \subseteq \mathcal{V}_G$ be open subsets with closed complements $W' \supset W$ respectively. Note that $\mathcal{C}_W^\oplus \subseteq \mathcal{C}_{W'}^\oplus$ and that therefore U -isomorphisms are U' -isomorphisms. Consider the induced localisation functor $\mathcal{T}(U) \rightarrow \mathcal{T}(U')$. For any pair of objects $M, N \in \mathcal{T}$, the induced homomorphism $\mathrm{Hom}_{\mathcal{T}(U)}(M, N) \rightarrow \mathrm{Hom}_{\mathcal{T}(U')}(M, N)$ gives a homomorphism

$$\mathrm{Hom}_{\mathcal{T}}(M \otimes F(W), N \otimes F(W)) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M \otimes F(W'), N \otimes F(W'))$$

by Corollary 5.3. This homomorphism can simply be described as follows

$$(M \otimes F(W) \xrightarrow{\alpha} N \otimes F(W)) \longmapsto (M \otimes F(W') \xrightarrow{\alpha \otimes F(W')} N \otimes F(W'))$$

using the identification $F(W) \otimes F(W') \cong F(W')$ of Theorem 5.1. This verification is left to the reader. See [21].

6. GLUING ARBITRARY MODULES

We now explain how to glue any pair of not necessarily finitely generated kG -modules. At this stage, always assuming $\mathcal{V}_G = U_1 \cup U_2$, we abandon $\mathrm{stmod}(kG)$ (in the right-hand

diagram below) and consider instead the left-hand commutative diagram of localisations of larger triangulated categories (see Notation 3.6):

$$\begin{array}{ccc} \mathcal{T} = \text{StMod}(kG) & \longrightarrow & \mathcal{T}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{T}(U_2) & \longrightarrow & \mathcal{T}(U_1 \cap U_2) \end{array} \quad \begin{array}{ccc} \mathcal{C} = \text{stmod}(kG) & \longrightarrow & \mathcal{C}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{C}(U_2) & \longrightarrow & \mathcal{C}(U_1 \cap U_2). \end{array}$$

The definition of a *gluing* $M \in \mathcal{T}$ of two objects $M_1 \in \mathcal{T}(U_1)$ and $M_2 \in \mathcal{T}(U_2)$ along an isomorphism $\sigma: M_1 \xrightarrow{\cong} M_2$ in $\mathcal{T}(U_1 \cap U_2)$ is exactly the same as in Definition 4.1, except of course that we allow M to live in the big category \mathcal{T} . Before proving existence and uniqueness of the gluing, let us unfold what happens to Rickard's idempotent modules in this situation.

Let us denote by $W_1 = \mathcal{V}_G \setminus U_1$ and $W_2 = \mathcal{V}_G \setminus U_2$ the closed complements of the two open subsets covering \mathcal{V}_G . By assumption, we have that $W_1 \cap W_2 = \emptyset$. Hence, $\mathcal{C}_{W_1 \cap W_2} = 0 = \mathcal{C}_{W_1 \cap W_2}^\oplus$. So we get from Theorem 5.1 that $E(W_1 \cap W_2) = 0$ and that $F(W_1 \cap W_2) = k$, as well as a Mayer–Vietoris distinguished triangle

$$k \xrightarrow{\begin{pmatrix} \varepsilon_1 \\ -\varepsilon_2 \end{pmatrix}} F(W_1) \oplus F(W_2) \xrightarrow{(\varepsilon_{12} \ \varepsilon_{21})} F(W_1 \cup W_2) \xrightarrow{\gamma} \Sigma k, \quad (6.1)$$

where the first two morphisms ε_i are the ε_{W_i} of Theorem 5.1 and where ε_{12} and ε_{21} are characterised by the commutativity of the following diagram:

$$\begin{array}{ccc} k & \xrightarrow{\varepsilon_1} & F(W_1) \\ \varepsilon_2 \downarrow & \searrow \varepsilon_{W_1 \cup W_2} & \downarrow \varepsilon_{12} \\ F(W_2) & \xrightarrow{\varepsilon_{21}} & F(W_1 \cup W_2). \end{array} \quad (6.2)$$

In the notation of [21, Def., p. 164], $\varepsilon_{12} = \varepsilon_{W_1, W_1 \cup W_2}$ and $\varepsilon_{21} = \varepsilon_{W_2, W_1 \cup W_2}$, or, using idempotence, $\varepsilon_{12} = \varepsilon_2 \otimes F(W_1)$ and $\varepsilon_{21} = \varepsilon_1 \otimes F(W_2)$. It will sometimes be convenient to abbreviate $\varepsilon := \varepsilon_{W_1 \cup W_2}$.

So, returning to our gluing problem, let M_1 and M_2 be objects of \mathcal{T} (thought of as objects of $\mathcal{T}(U_1)$ and $\mathcal{T}(U_2)$ respectively) and let $\sigma: M_1 \xrightarrow{\cong} M_2$ be an isomorphism in $\mathcal{T}(U_1 \cap U_2)$. By Corollary 5.3, there exists an isomorphism $\alpha: M_1 \otimes F(W_1 \cup W_2) \xrightarrow{\cong} M_2 \otimes F(W_1 \cup W_2)$ in \mathcal{T} such that the following diagram of isomorphisms commutes in $\mathcal{T}(U_1 \cap U_2)$:

$$\begin{array}{ccc} M_1 & \xrightarrow{\sigma} & M_2 \\ M_1 \otimes \varepsilon \downarrow & & \downarrow M_2 \otimes \varepsilon \\ M_1 \otimes F(W_1 \cup W_2) & \xrightarrow{\alpha} & M_2 \otimes F(W_1 \cup W_2). \end{array} \quad (6.3)$$

Using this isomorphism α , we can now give our main construction.

Definition 6.1. Let $M_1, M_2 \in \mathcal{T}$ and let $\alpha: M_1 \otimes F(W_1 \cup W_2) \xrightarrow{\cong} M_2 \otimes F(W_1 \cup W_2)$ be an isomorphism in \mathcal{T} . Consider the following morphism:

$$(M_1 \otimes F(W_1)) \oplus (M_2 \otimes F(W_2)) \xrightarrow{\begin{pmatrix} \alpha \circ (M_1 \otimes \varepsilon_{12}) & M_2 \otimes \varepsilon_{21} \end{pmatrix}} M_2 \otimes F(W_1 \cup W_2)$$

and complete it into a distinguished triangle:

$$X_\alpha \xrightarrow{\begin{pmatrix} \varepsilon_1^\alpha \\ -\varepsilon_2^\alpha \end{pmatrix}} (M_1 \otimes F(W_1)) \oplus (M_2 \otimes F(W_2)) \xrightarrow{\begin{pmatrix} \alpha \circ (M_1 \otimes \varepsilon_{12}) & M_2 \otimes \varepsilon_{21} \end{pmatrix}} M_2 \otimes F(W_1 \cup W_2) \xrightarrow{\gamma^\alpha} \Sigma X_\alpha$$

for some object $X_\alpha \in \mathcal{T}$ and morphisms $\varepsilon_1^\alpha, \varepsilon_2^\alpha$ and γ^α as above. (In this notation, we only indicate dependence on α but not on M_1 and M_2 , nor on W_1 and W_2 , for obvious reasons.)

Note that the module X_α is only well-defined up to (non-unique) isomorphism.

Remark 6.2. We have the following weak pullback (Definition 3.4):

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\varepsilon_1^\alpha} & M_1 \otimes F(W_1) \\ \varepsilon_2^\alpha \downarrow & & \downarrow \alpha \circ (M_1 \otimes \varepsilon_{12}) \\ M_2 \otimes F(W_2) & \xrightarrow{M_2 \otimes \varepsilon_{21}} & M_2 \otimes F(W_1 \cup W_2). \end{array} \quad (6.4)$$

Indeed, this characterises X_α , as we shall see in Lemma 6.5. The reader might prefer the following more symmetric ‘‘square’’:

$$\begin{array}{ccccc} & & X_\alpha & & \\ & \swarrow \varepsilon_1^\alpha & & \searrow \varepsilon_2^\alpha & \\ M_1 \otimes F(W_1) & & & & M_2 \otimes F(W_2) \\ & \searrow M_1 \otimes \varepsilon_{12} & & \swarrow M_2 \otimes \varepsilon_{21} & \\ & & M_1 \otimes F(W_1 \cup W_2) & \xrightarrow[\sim]{\alpha} & M_2 \otimes F(W_1 \cup W_2) \end{array}$$

Lemma 6.3. Recall the notion of U -isomorphism from Definition 3.2.

- (1) The morphism ε_{12} is a U_2 -isomorphism and ε_{21} is a U_1 -isomorphism.
- (2) For every objects $M_1, M_2 \in \mathcal{T}$ and every isomorphism $\alpha: M_1 \otimes F(W_1 \cup W_2) \xrightarrow{\cong} M_2 \otimes F(W_1 \cup W_2)$, the morphism ε_i^α is a U_i -isomorphism for $i = 1, 2$.

Proof. We know that ε_i is a U_i -isomorphism by Theorem 5.1 and the statement for ε_{12} and ε_{21} follows by Proposition 3.3 and the Mayer–Vietoris triangle (6.1). The second part of the statement is a consequence of the same Proposition and the distinguished triangle of Definition 6.1. Recall from Definition 3.1 that $M_1 \otimes \varepsilon_{12}$ is still a U_1 -isomorphism. \square

Proposition 6.4. We have $\mathcal{C}_{W_1}^\oplus \cap \mathcal{C}_{W_2}^\oplus = 0$. In particular, if a morphism $f: L \rightarrow M$ in $\text{StMod}(kG)$ is both a U_1 -isomorphism and a U_2 -isomorphism, then f is an isomorphism.

Proof. Let $N \in \mathcal{C}_{W_i}^\oplus$ for $i = 1, 2$. Then $N \otimes F(W_i) = 0$. (This follows from Theorem 5.1 or can be found explicitly as [21, Proposition 5.15].) A fortiori, $N \otimes F(W_1 \cup W_2) = N \otimes F(W_1) \otimes F(W_2) = 0$. But then, tensoring N with the Mayer–Vietoris triangle (6.1), we see

that $N = N \otimes k = 0$, as claimed. This proves the first statement. The second follows from the first since $\text{cone}(f) \in \mathcal{C}_{W_1}^\oplus \cap \mathcal{C}_{W_2}^\oplus$. \square

Lemma 6.5. *Let $M_1, M_2 \in \mathcal{T}$ and let $\alpha: M_1 \otimes F(W_1 \cup W_2) \xrightarrow{\cong} M_2 \otimes F(W_1 \cup W_2)$ be an isomorphism. Consider a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & M_1 \otimes F(W_1) \\ f_2 \downarrow & & \downarrow \alpha \circ (M_1 \otimes \varepsilon_{12}) \\ M_2 \otimes F(W_2) & \xrightarrow{M_2 \otimes \varepsilon_{21}} & M_2 \otimes F(W_1 \cup W_2). \end{array}$$

Then the following conditions are equivalent:

- (1) The above square is a weak pullback (see Definition 3.4).
- (2) f_i is a U_i -isomorphism for $i = 1, 2$.

When these conditions hold, the module X is isomorphic to the X_α of Definition 6.1.

Proof. If the square is a weak pullback, Proposition 3.3 insures that f_1 is a U_1 -isomorphism since ε_{21} is a U_1 -isomorphism by Lemma 6.3. Similarly, f_2 is a U_2 -isomorphism.

Conversely, suppose that f_i is a U_i -isomorphism for $i = 1, 2$, and construct the weak pullback X_α (see Remark 6.2):

$$\begin{array}{ccccc} X & & \xrightarrow{f_1} & & M_1 \otimes F(W_1) \\ & \searrow f & & \nearrow \varepsilon_1^\alpha & \downarrow \alpha \circ (M_1 \otimes \varepsilon_{12}) \\ & & X_\alpha & \xrightarrow{\varepsilon_1^\alpha} & M_1 \otimes F(W_1) \\ & \searrow f_2 & \varepsilon_2^\alpha \downarrow & & \downarrow \alpha \circ (M_1 \otimes \varepsilon_{12}) \\ & & M_2 \otimes F(W_2) & \xrightarrow{M_2 \otimes \varepsilon_{21}} & M_2 \otimes F(W_1 \cup W_2). \end{array}$$

Since the outer diagram commutes, there exists a corner morphism $f: X \rightarrow X_\alpha$ making the whole diagram commute. This is the weak pullback property. By Lemma 6.3, ε_i^α is a U_i -isomorphism for $i = 1, 2$. By two-out-of-three, we see that f is a U_i -isomorphism as well, for $i = 1, 2$. Hence, by Proposition 6.4, f is an isomorphism. \square

Theorem 6.6. *Let $M_1, M_2 \in \mathcal{T}$ and let $\alpha: M_1 \otimes F(W_1 \cup W_2) \xrightarrow{\cong} M_2 \otimes F(W_1 \cup W_2)$ be an isomorphism, which corresponds, via Corollary 5.3, to an isomorphism $\sigma: M_1 \xrightarrow{\cong} M_2$ in $\mathcal{T}(U_1 \cap U_2)$. Then the object X_α constructed in Definition 6.1 is a gluing of M_1 and M_2 along the isomorphism σ . Moreover, this gluing is unique up to isomorphism in \mathcal{T} .*

Proof. Recall from Corollary 5.3 that in $\mathcal{T}(U_1 \cap U_2)$, we have $\sigma = (M_2 \otimes \varepsilon)^{-1} \circ \alpha \circ (M_1 \otimes \varepsilon)$, where $\varepsilon = \varepsilon_{W_1 \cup W_2}$, as presented in Diagram (6.3).

Let us first check that X_α is indeed a gluing. For $i = 1, 2$, define the isomorphisms $\sigma_i: X_\alpha \xrightarrow{\cong} M_i$ in $\mathcal{T}(U_i)$ by $\sigma_i := (M_i \otimes \varepsilon_i)^{-1} \circ \varepsilon_i^\alpha$

$$X_\alpha \begin{array}{c} \xrightarrow{\varepsilon_i^\alpha} \\ \xrightarrow{\sigma_i} \\ \xleftarrow{M_i \otimes \varepsilon_i} \end{array} M_i \otimes F(W_i) \begin{array}{c} \xleftarrow{\sigma_i} \\ \xleftarrow{M_i \otimes \varepsilon_i} \end{array} M_i.$$

Recall from Lemma 6.3 that ε_i and ε_i^α are U_i -isomorphisms, hence $M_i \otimes \varepsilon_i$ as well. Recall that we have two commutative Diagrams (6.2) and (6.4) in \mathcal{T} which are respectively:

$$(6.2) : \begin{array}{ccc} k & \xrightarrow{\varepsilon_1} & F(W_1) \\ \varepsilon_2 \downarrow & \searrow \varepsilon & \downarrow \varepsilon_{12} \\ F(W_2) & \xrightarrow{\varepsilon_{21}} & F(W_1 \cup W_2) \end{array} \quad \text{and} \quad (6.4) : \begin{array}{ccc} X_\alpha & \xrightarrow{\varepsilon_1^\alpha} & M_1 \otimes F(W_1) \\ \varepsilon_2^\alpha \downarrow & & \downarrow \alpha \circ (M_1 \otimes \varepsilon_{12}) \\ M_2 \otimes F(W_2) & \xrightarrow{M_2 \otimes \varepsilon_{21}} & M_2 \otimes F(W_1 \cup W_2) \end{array}$$

and all morphisms in sight are $(U_1 \cap U_2)$ -isomorphisms by Lemma 6.3. Now we compute in $\mathcal{T}(U_1 \cap U_2)$ using the commutativity of the above squares:

$$\begin{aligned} \sigma \sigma_1 &\stackrel{\text{def}}{=} (M_2 \otimes \varepsilon)^{-1} \alpha (M_1 \otimes \varepsilon) (M_1 \otimes \varepsilon_1)^{-1} \varepsilon_1^\alpha \stackrel{(6.2)}{=} (M_2 \otimes \varepsilon)^{-1} \alpha (M_1 \otimes \varepsilon_{12}) \varepsilon_1^\alpha = \\ &\stackrel{(6.4)}{=} (M_2 \otimes \varepsilon)^{-1} (M_2 \otimes \varepsilon_{21}) \varepsilon_2^\alpha \stackrel{(6.2)}{=} (M_2 \otimes \varepsilon_2)^{-1} \varepsilon_2^\alpha \stackrel{\text{def}}{=} \sigma_2. \end{aligned}$$

This proves that $(X_\alpha, \sigma_1, \sigma_2)$ is a gluing of M_1 and M_2 along σ .

Let us now turn to uniqueness. Let (X, τ_1, τ_2) be another gluing of M_1 and M_2 along σ , that is, $\tau_i: X \xrightarrow{\cong} M_i$ in $\mathcal{T}(U_i)$ for $i = 1, 2$ and $\sigma \circ \tau_1 = \tau_2$ in $\mathcal{T}(U_1 \cap U_2)$. Consider the morphisms $(M_i \otimes \varepsilon_i) \circ \tau_i: X \rightarrow M_i \otimes F(W_i)$ in $\mathcal{T}(U_i)$ for $i = 1, 2$. By Proposition 5.2, there exist two morphisms $f_i: X \rightarrow M_i \otimes F(W_i)$, $i = 1, 2$, which give the above morphisms $(M_i \otimes \varepsilon_i) \circ \tau_i$ under localisation, i.e., such that the following diagram commutes in $\mathcal{T}(U_i)$:

$$X \begin{array}{c} \xrightarrow{f_i} \\ \xrightarrow{\tau_i} \\ \xleftarrow{M_i \otimes \varepsilon_i} \end{array} M_i \otimes F(W_i) \begin{array}{c} \xleftarrow{\tau_i} \\ \xleftarrow{M_i \otimes \varepsilon_i} \end{array} M_i.$$

In particular, it is immediate that f_i is a U_i -isomorphism. We now want to apply Lemma 6.5 to the triple (X, f_1, f_2) :

$$\begin{array}{ccc} X & \xrightarrow{f_1} & M_1 \otimes F(W_1) \\ f_2 \downarrow & & \downarrow \alpha \circ (M_1 \otimes \varepsilon_{12}) \\ M_2 \otimes F(W_2) & \xrightarrow{M_2 \otimes \varepsilon_{21}} & M_2 \otimes F(W_1 \cup W_2). \end{array}$$

We have already checked condition (2) of that Lemma and it only remains to check that the above square really commutes. To see this, note that the lower right object, the target of both compositions, is of the form $(\dots) \otimes F(W_1 \cup W_2)$. So, using Proposition 5.3, it is enough to check the commutativity of that square in the localisation $\mathcal{T}(U_1 \cap U_2)$. There, it

becomes easy, for it exactly amounts to the condition $\sigma\tau_1 = \tau_2$, as can be readily verified using Diagram (6.2) again. \square

Corollary 6.7. *Let $M \in \mathcal{T} = \mathbf{StMod}(kG)$ and suppose that M is finitely generated on U_1 and U_2 , that is, M is isomorphic in $\mathcal{T}(U_i)$ to an object of $\mathcal{C}(U_i) \subseteq \mathcal{T}(U_i)$ for $i = 1, 2$. Then M is finitely generated, that is, M is isomorphic to an object of $\mathcal{C} = \mathbf{stmod}(kG)$.*

Proof. Consider $M_i \in \mathcal{C}(U_i)$ and $\sigma_i: M \xrightarrow{\cong} M_i$ in $\mathcal{C}(U_i)$ for $i = 1, 2$. Define the isomorphism $\sigma = \sigma_2 \sigma_1^{-1}: M_1 \xrightarrow{\cong} M_2$ in $\mathcal{C}(U_1 \cap U_2)$ – here we use Proposition 3.8. Then, obviously, M is the gluing of M_1 and M_2 along σ in \mathcal{T} . As already mentioned, we know from [3, Cor. 5.10] that the gluing is possible in \mathcal{C} , that is, there exists a gluing $M' \in \mathcal{C}$ of M_1 and M_2 along σ . Since the gluing is unique in the big category \mathcal{T} , we must have $M \cong M'$. \square

7. A GLUING CONSTRUCTION OF ENDOTRIVIAL MODULES

We now unfold the general gluing construction of Section 6 in the special case of

$$M_1 = M_2 = k.$$

The outcome, in that case, is an endotrivial (finitely generated) module. As before, we assume that we have an open covering $\mathcal{V}_G = U_1 \cup U_2$ of the projective support variety and we denote by $W_i = \mathcal{V}_G \setminus U_i$ the closed complements $i = 1, 2$. Here, Definition 6.1 becomes:

Definition 7.1. Let $\alpha \in \mathbf{Aut}_{\mathcal{T}}(F(W_1 \cup W_2))$ be an automorphism of $F(W_1 \cup W_2)$ in $\mathcal{T} = \mathbf{StMod}(kG)$, i.e., a unit in $\mathbf{End}_{kG}(F(W_1 \cup W_2))$. Consider the morphism

$$F(W_1) \oplus F(W_2) \xrightarrow{(\alpha \circ \varepsilon_{12} \quad \varepsilon_{21})} F(W_1 \cup W_2)$$

which differs from the middle map of the Mayer-Vietoris triangle (6.1), only in that we twist the first component by the automorphism α . Completing this morphism to a distinguished triangle defines a module X_α and morphisms $\varepsilon_1^\alpha, \varepsilon_2$ and γ^α as follows:

$$X_\alpha \xrightarrow{\begin{pmatrix} \varepsilon_1^\alpha \\ -\varepsilon_2^\alpha \end{pmatrix}} F(W_1) \oplus F(W_2) \xrightarrow{(\alpha \varepsilon_{12} \quad \varepsilon_{21})} F(W_1 \cup W_2) \xrightarrow{\gamma^\alpha} \Sigma X_\alpha. \quad (7.1)$$

As before, the module X_α is only well-defined up to (non-unique) isomorphism. We shall only be interested in the isomorphism class of X_α in $\mathbf{StMod}(kG)$. If $\alpha: k \rightarrow k$ is the identity then the Mayer-Vietoris triangle (6.1) shows that $X_{\text{Id}} \cong k$. It should also be pointed out that the definition of X_α depends on the ordering of the two disjoint closed subsets W_1 and W_2 of the support variety. In 7.6 we see what happens if we interchange W_1 and W_2 .

Remark 7.2. As in Remark 6.2, we have a weak pullback:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\varepsilon_1^\alpha} & F(W_1) \\ \varepsilon_2^\alpha \downarrow & & \downarrow \alpha \varepsilon_{12} \\ F(W_2) & \xrightarrow{\varepsilon_{21}} & F(W_1 \cup W_2), \end{array} \quad (7.2)$$

which characterises X_α , by Lemma 6.5. Taking two automorphisms α and β , we can tensor the above square with the similar square for β . Using idempotence (Theorem 5.1), we get

$$\begin{array}{ccc} X_\alpha \otimes X_\beta & \xrightarrow{\varepsilon_1^\alpha \otimes \varepsilon_1^\beta} & F(W_1) \\ \varepsilon_2^\alpha \otimes \varepsilon_2^\beta \downarrow & & \downarrow \alpha\beta\varepsilon_{12} \\ F(W_2) & \xrightarrow{\varepsilon_{21}} & F(W_1 \cup W_2). \end{array}$$

By Lemma 6.5 (2), the latter square is a weak pullback and therefore $X_\alpha \otimes X_\beta \cong X_{\alpha\beta}$. In particular $X_\alpha \otimes X_{\alpha^{-1}} \cong k$ and Theorem 2.1 forces X_α to belong to $\mathbf{stmod}(kG)$, i.e., X_α is a finitely generated endotrivial module. We give another proof of these facts below.

Theorem 7.3. *Let $\alpha \in \underline{\mathbf{End}}_{kG}(F(W_1 \cup W_2))$ be an automorphism in $\mathbf{StMod}(kG)$. Then the module X_α of Definition 7.1 is isomorphic in $\mathbf{StMod}(kG)$ to a finite dimensional endotrivial module, that is, $X_\alpha \in \mathcal{C} = \mathbf{stmod}(kG)$.*

Moreover, if $\sigma \in \mathbf{Aut}_{\mathcal{C}(U_1 \cap U_2)}(k)$ is the automorphism of k over $U_1 \cap U_2$ corresponding to α (see Corollary 5.4), then the module $X_\alpha \in \mathbf{stmod}(kG)$ is a gluing (Definition 4.1) of two copies of k along the isomorphism $\sigma: k \xrightarrow{\cong} k$ in $\mathcal{C}(U_1 \cap U_2)$.

Proof. We already know by Theorem 6.6, applied to $M_1 = M_2 = k$, that X_α is a gluing of two copies of k along σ in the big category \mathcal{J} . Corollary 6.7 tells us that $X_\alpha \in \mathbf{stmod}(kG)$. It is endotrivial because it is locally endotrivial. That is, the evaluation map $(X_\alpha)^* \otimes X_\alpha \rightarrow k$ is an isomorphism in $\mathcal{C}(U_i)$ for $i = 1, 2$, hence is an isomorphism in \mathcal{C} (its cone has empty support). Another proof of the latter fact was given in Remark 7.2 above. (Alternatively, see [3, Lem. 6.2].) \square

We now have the following dictionary with the terminology of [3]:

Corollary 7.4. *Consider $T_k(G) = \mathbf{Pic}(\mathcal{C})$, the group of finitely generated endotrivial kG -modules, i.e., the group of invertible objects in $\mathcal{C} = \mathbf{stmod}(kG)$ with respect to \otimes . Consider the map $\xi: \mathbf{Aut}_{\mathcal{J}}(F(W_1 \cup W_2)) \rightarrow T_k(G)$ given by the above construction, $\alpha \mapsto X_\alpha$. Consider the homomorphism $\delta: \mathbb{G}_m(U_1 \cap U_2) = \mathbf{Aut}_{\mathcal{C}(U_1 \cap U_2)}(k) \rightarrow \mathbf{Pic}(\mathcal{C})$ of Theorem 4.2. Then the two maps ξ and δ are equal. More precisely, the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{Aut}_{\mathcal{J}}(F(W_1 \cup W_2)) & \xrightarrow[\text{(Cor. 5.4)}]{\sim} & \mathbb{G}_m(U_1 \cap U_2) \\ \xi \downarrow & & \downarrow \delta \\ T_k(G) & \xlongequal{\quad} & \mathbf{Pic}(\mathcal{C}). \end{array}$$

In particular, if α and β are two invertible elements in $\underline{\mathbf{End}}_{kG}(F(W_1 \cup W_2))$ then

$$X_\alpha \otimes X_\beta \cong X_{\alpha\beta}.$$

Proof. This is simply a condensed form of the previous results. Note that ξ is a homomorphism because δ is already known to be one, hence $X_\alpha \otimes X_\beta \cong X_{\alpha\beta}$. (We sketched a direct proof of the latter in Remark 7.2.) \square

Corollary 7.5. *Let $\alpha \in \underline{\text{End}}_{kG}(F(W_1 \cup W_2))$ be an automorphism. Then $X_\alpha \cong k$ is trivial if and only if there exists automorphisms $\alpha_i \in \underline{\text{End}}_{kG}(F(W_i))$ for $i = 1, 2$ such that $\alpha = \alpha_1 \otimes \alpha_2$ under the identification $F(W_1 \cup W_2) \cong F(W_1) \otimes F(W_2)$.*

Proof. Transcribe in modular representation theoretic terms the exactness of the sequence

$$\cdots \longrightarrow \mathbb{G}_m(U_1) \oplus \mathbb{G}_m(U_2) \longrightarrow \mathbb{G}_m(U_1 \cap U_2) \xrightarrow{\delta} \text{Pic}(\text{stmod}(kG))$$

established in [3, Thm. 6.7] as part of the Mayer-Vietoris long exact sequence. \square

Remark 7.6. The definition of X_α (Definition 7.1) is not symmetric in the two closed subsets W_1 and W_2 and this might lead to some confusion. Strictly speaking, we should write $X_\alpha = X(\alpha, W_1, W_2)$. Switching the order of W_1 and W_2 inverts the module X_α (i.e., gives the dual $(X_\alpha)^*$ instead). This can be easily checked, for instance from the gluing property of Theorem 7.3, which says that the following left-hand diagram commutes in $\mathcal{C}(U_1 \cap U_2)$:

$$\begin{array}{ccc} & & k \\ & \nearrow^{\sigma_1} & \downarrow \sigma \\ X_\alpha & & k \\ & \searrow_{\sigma_2} & \downarrow \sigma^{-1} \\ & & k \end{array} \implies \begin{array}{ccc} & & k \\ & \nearrow^{\sigma_2} & \downarrow \sigma^{-1} \\ X_\alpha & & k \\ & \searrow_{\sigma_1} & \downarrow \sigma^{-1} \\ & & k \end{array}$$

Hence the right-hand diagram commutes as well and by Theorem 7.3 again but applied to (W_2, W_1) , we obtain $X(\alpha^{-1}, W_2, W_1) = X_\alpha$ which implies $X(\alpha, W_2, W_1) = X_{\alpha^{-1}} = (X_\alpha)^*$.

Remark 7.7. Of course, the definition of X_α given in Definition 7.1 also makes sense if $\alpha \in \underline{\text{End}}_{kG}(F(W_1 \cup W_2))$ is a non-invertible endomorphism. The problem is that the module X_α will not be endotrivial in general. Take for instance $\alpha = 0$. Then we have $(\alpha \varepsilon_{12} \ \varepsilon_{21}) = (0 \ \varepsilon_{21}): F(W_1) \oplus F(W_2) \rightarrow F(W_1 \cup W_2)$, and the defining triangle (7.1) becomes:

$$F(W_1) \oplus (E(W_1) \otimes F(W_2)) \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \eta_{W_1} \otimes 1 \end{pmatrix}} F(W_1) \oplus F(W_2) \xrightarrow{(0 \ \varepsilon_{21})} F(W_1 \cup W_2) \xrightarrow{\begin{pmatrix} 0 \\ \theta_{W_1} \otimes 1 \end{pmatrix}} \Sigma(\dots).$$

To see that this triangle is distinguished, apply $-\otimes F(W_2)$ to the original triangle for $E(W_1)$ and $F(W_1)$, from Theorem 5.1 (1), and then add the trivial triangle

$$F(W_1) \xrightarrow{1} F(W_1) \longrightarrow 0 \longrightarrow \Sigma F(W_1).$$

So, the module X_0 is $F(W_1) \oplus (E(W_1) \otimes F(W_2))$ which is not even in $\text{stmod}(kG)$ in general.

Remark 7.8. There is an extreme situation where our construction produces an endotrivial module for any endomorphism α , even the most trivial $\alpha = 0$. Namely, this happens if $W_1 = \emptyset$. Indeed, in that case, $F(W_1) = k$ and the weak pullback (7.2) becomes:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\varepsilon_1^\alpha} & k \\ \varepsilon_2^\alpha \downarrow & & \downarrow \forall \alpha \\ F(W_2) & \xlongequal{\quad} & F(W_2) \end{array}$$

which forces $\varepsilon_1^\alpha: X_\alpha \xrightarrow{\cong} k$ to be an isomorphism. This rather trivial remark will be useful at the end of the paper.

8. FIRST EXAMPLE: RANK TWO

Suppose that ζ_1 and ζ_2 are elements of $H^n(G, k)$ such that $\mathcal{V}_G(\zeta_1) \cap \mathcal{V}_G(\zeta_2) = \emptyset$. Note that this implies that $H^n(G, k)$ is a finitely generated module over $k[\zeta_1, \zeta_2]$, so that it has Krull dimension two, which by Quillen's Dimension Theorem forces the group G to have p -rank two. If $p > 2$, then it also requires that ζ_1 and ζ_2 have even degree, as otherwise ζ_1 and ζ_2 are nilpotent. In particular, ζ_1 and ζ_2 must commute with each other. Letting $W_1 = \mathcal{V}_G(\zeta_1)$ and $W_2 = \mathcal{V}_G(\zeta_2)$, we are in precisely the situation of Section 7. Note that $W_1 \cup W_2 = \mathcal{V}_G(\zeta_1 \zeta_2)$.

The endomorphism ring $\underline{\text{End}}_{kG}(F(W_1))$ consists of the degree zero elements of the localisation $H^*(G, k)[\zeta_1^{-1}]$ by Proposition 5.6, and similarly for $F(W_2)$ and $H^*(G, k)[\zeta_2^{-1}]$ and for $F(W_1 \cup W_2)$ and $H^*(G, k)[\zeta_1^{-1}\zeta_2^{-1}]$.

The element $\zeta_2/\zeta_1 \in (H^*(G, k)[\zeta_1^{-1}])^0 \cong \underline{\text{End}}_{kG}(F(W_1))$ is not invertible, but it becomes invertible in $H^*(G, k)[\zeta_1^{-1}\zeta_2^{-1}]$ with inverse ζ_1/ζ_2 . So it is an automorphism of $F(W_1 \cup W_2)$ and is a candidate for α in the construction of our module X_α (Definition 7.1). Note however that we can consider other automorphisms of $F(W_1 \cup W_2)$. This particular $\alpha = \zeta_1/\zeta_2$ is in some sense the "trivial" choice since ζ_1 and ζ_2 are precisely the defining equations of $W_1 = \mathcal{V}_G(\zeta_1)$ and $W_2 = \mathcal{V}_G(\zeta_2)$. In Section 9, we shall consider an example with another α .

Proposition 8.1. *With the above notation, the module X_{ζ_1/ζ_2} is isomorphic to $\Sigma^n k = \Omega^{-n}k$.*

Proof. It is clear that each $\zeta_i: k \rightarrow \Sigma^n k$ is a U_i -isomorphism (Definition 3.2) since its cone is a shift of the Carlson module L_{ζ_i} which has support exactly W_i (see [4]). Let $\sigma_i = (\zeta_i)^{-1}: \Sigma^n k \xrightarrow{\cong} k$ be the inverse isomorphism in the localisation $\mathcal{C}(U_i)$. In the final localisation $\mathcal{C}(U_1 \cap U_2)$, the following diagram obviously commutes:

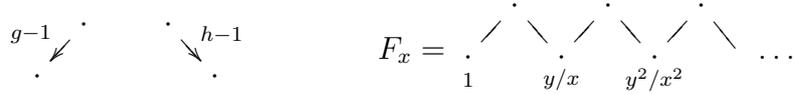
$$\begin{array}{ccc} & k & \\ \sigma_1 \nearrow & & \downarrow \zeta_1/\zeta_2 \\ \Sigma^n k & & k \\ \sigma_2 \searrow & & \end{array}$$

In the language of [3], this means that $\Sigma^n k$ is the gluing of two copies of k along the automorphism $\zeta_1/\zeta_2 \in \text{Aut}_{\mathcal{C}(U_1 \cap U_2)}(k)$, i.e., we have $X_\alpha = \Sigma^n k$ by Theorem 7.3. \square

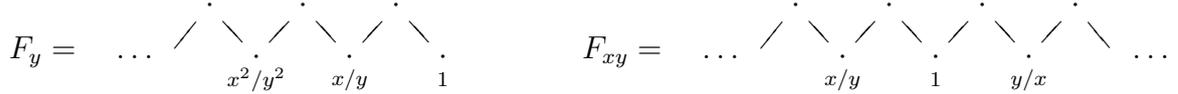
It is also possible to prove the previous proposition by describing the idempotent modules, $F(W_1)$, $F(W_2)$ and $F(W_1 \cup W_2)$, as colimits, giving the various morphisms $\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \varepsilon_{21}$ as maps on the colimits, and then making the distinguished triangle (7.1) explicit. However, the proof using the gluing technique that we have given here is considerably shorter.

Example 8.2. We unfold our construction of the endotrivial modules with a very explicit example in the case of the fours group $G = \langle g, h \rangle \cong (\mathbb{Z}/2)^2$ over a field k of characteristic two. We have $H^*(G, k) = k[x, y]$ with $\deg(x) = \deg(y) = 1$. Here, the basis x, y of $H^1(G, k)$ is dual to the basis $g - 1, h - 1$ of $H_1(G, k) \cong J(kG)/J^2(kG)$. The support variety is $\mathcal{V}_G = \text{Proj } k[x, y] = \mathbb{P}^1(k)$. Let $W_1 = \mathcal{V}_G(x)$, $W_2 = \mathcal{V}_G(y)$, the zero loci of x and y respectively. We use the open covering by the two affine sets $U_i = \mathcal{V}_G \setminus W_i$, $i = 1, 2$ defined by x and y , with intersection $U_1 \cap U_2 = \mathcal{V}_G \setminus \mathcal{V}_G(xy)$ and union $U_1 \cup U_2 = \mathcal{V}_G$. The module

$F_x = F(W_1)$ is represented by the following diagram:



Here, we have labeled basis elements of the socle of F_x with the corresponding elements of $\underline{\text{Hom}}_{kG}(k, F_x)$, namely degree zero elements of $H^*(G, k)[x^{-1}]$. Similarly, we have

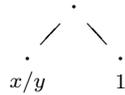


The maps $\varepsilon_{12}: F_x \rightarrow F_{xy}$ and $\varepsilon_{21}: F_y \rightarrow F_{xy}$ correspond to the obvious inclusions of diagrams.

The endomorphism ring of F_{xy} consists of the degree zero elements of $H^*(G, k)[x^{-1}y^{-1}]$ by Proposition 5.6. So x/y is an automorphism with inverse y/x . The action of x/y on F_{xy} is a shift one place to the left, while y/x is a shift one place to the right. Let $\alpha = x/y$. Then we have a diagram

$$\begin{array}{ccc}
 & F_x & \\
 & \downarrow \scriptstyle{(x/y)\varepsilon_{12}} & \\
 F_y & \xrightarrow{\scriptstyle{\varepsilon_{21}}} & F_{xy}
 \end{array}$$

Since the sum of the two maps is surjective, the weak pullback of this diagram is the same as the ordinary pullback, namely the submodule corresponding to the intersection of the subdiagrams. This gives the diagram



for $X_\alpha \cong \Omega^{-1}(k)$, in accordance with Proposition 8.1. On the other hand, if we use the endomorphism $\alpha^{-1} = y/x$, then the subdiagrams do not intersect, and the weak pullback is not the same as the pullback. We must add a projective to make the sum of the maps surjective, and then take the pullback to obtain $X_{\alpha^{-1}} \cong \Omega(k)$.

9. SECOND EXAMPLE: THE DIHEDRAL GROUP D_8

In this section we discuss the example of the dihedral group $G = D_8$ over a field k of characteristic two. The arguments that we use here are a model for what appears in Section 10.

The cohomology ring of G has the form

$$H^*(G, k) = k[x, y, z]/(xy)$$

where $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. Consider $\zeta_1 = x^2 + y^2 + z$ and $\zeta_2 = z$ in $H^2(G, k)$ and define $W_1 = \mathcal{V}_G(\zeta_1)$ and $W_2 = \mathcal{V}_G(\zeta_2)$ as in Section 8. We claim that $W_1 \cap W_2 = \emptyset$. Indeed, a homogenous prime ideal \mathfrak{p} of $H^*(G, k) = k[x, y, z]/(xy)$ containing $x^2 + y^2$ and z

necessarily contains x, y and z (it contains $xy = 0$ hence x or y hence both since it contains $x^2 + y^2$). But then \mathfrak{p} contains the maximal ideal (x, y, z) which is excluded in $\text{Proj}(H^*(G, k))$. So, we have the open covering necessary for our construction of endotrivial modules of the form X_α as in Section 7

$$\mathcal{V}_G = U_1 \cup U_2$$

where $U_i = \mathcal{V}_G \setminus W_i$ for $i = 1, 2$. Now we want to produce an automorphism α of $F(W_1 \cup W_2)$ in a more subtle way than in Section 8, that is, different from $\zeta_1/\zeta_2 = (x^2 + y^2 + z)/z$.

Because $(x^2 + z)(y^2 + z) = (x^2 + y^2 + z)z$ in $H^*(G, k)$, we have that

$$\frac{x^2 + z}{x^2 + y^2 + z} \cdot \frac{y^2 + z}{z} = 1$$

in $H^*(G, k)[(x^2 + y^2 + z)^{-1}z^{-1}] = H^*(G, k)[(\zeta_1\zeta_2)^{-1}]$. So $(y^2 + z)/z$ is invertible in this localisation with inverse $(x^2 + z)/(x^2 + y^2 + z)$.

As before we use the notation $F_u = F(\mathcal{V}_G(u))$, where $u \in H^*(G, k)$. We set $\alpha = (y^2 + z)/z$, so that X_α is the weak pullback

$$\begin{array}{ccc} X_\alpha & \longrightarrow & F_{x^2+y^2+z} \\ \downarrow & & \downarrow \frac{y^2+z}{z} \varepsilon_{12} \\ F_z & \xrightarrow{\varepsilon_{21}} & F_{(x^2+y^2+z)z} \end{array}$$

We can characterize the endotrivial module X_α by restricting it to the two subgroups isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and by applying Proposition 8.1. On one of these subgroups, H_1 , x restricts to zero and y does not, and on the other, H_2 , y restricts to zero and x does not. The element z restricts to the product of the remaining two nonzero elements of $H^1(H_1, \mathbb{F}_2) \cong \mathbb{F}_2^2$ and of $H^1(H_2, \mathbb{F}_2) \cong \mathbb{F}_2^2$. So the restriction of α to H_2 is the identity element, while on H_1 it is a ratio of two degree-two elements with no common factor, and is the same as the restriction of $\zeta_1/\zeta_2 = (x^2 + y^2 + z)/z$. So $X_\alpha \downarrow_{H_2} \cong k$ while $X_\alpha \downarrow_{H_1}$ is isomorphic to $\Omega^{-2}k$ by Proposition 8.1. This module X_α is one of two well known five dimensional endotrivial modules, and has the following diagram:

$$\begin{array}{c} \cdot \\ \swarrow \quad \searrow \\ \cdot \quad \cdot \\ \parallel \quad \parallel \\ \cdot \quad \cdot \\ \vdots \quad \vdots \\ \cdot \quad \cdot \end{array}$$

See [9] for more details on the diagrams.

10. THE RANK OF THE GROUP OF ENDOTRIVIAL MODULES

Assume throughout this section that the p -rank of G is at least 2. We demonstrate that the construction of Section 7 yields a sufficiency of modules to generate a subgroup of finite index in the group $T(G)$ of endotrivial kG -modules. Hence, we have another proof of the rank of the torsion free subgroup of $T(G)$. The rank was first obtained in [1], though Alperin's original proof was only meant to apply to the case that G is a p -group. The determination

of the rank of $T(G)$ in [11] is valid for all finite groups. The proof given here follows roughly the lines of that one, and like that proof, it relies heavily on the context for the problem laid out and proved in [1]. Specifically, we have the following.

By [1], there exists a collection E_1, \dots, E_n of elementary abelian subgroups with the property that

- (1) Every E_i has p -rank 2,
- (2) If G has p -rank 2, then the subgroups E_1, \dots, E_n are a complete set of representatives of the conjugacy classes of maximal elementary abelian p -subgroups of G , and
- (3) If G has p -rank greater than 2, then E_1, \dots, E_{n-1} is a complete set of representatives of the conjugacy classes of maximal elementary abelian p -subgroups of G of rank 2, E_n is normal in a Sylow p -subgroup of G , and E_n is conjugate to a subgroup of any maximal elementary abelian p -subgroup of G that has p -rank greater than 2.

Notice here that if G has no maximal elementary abelian p -subgroup of p -rank 2, then $n = 1$.

The above result is explained in more detail in [12, (2.2)]. While it is proved for p -groups, the extensions to general finite groups is straightforward.

Proposition 10.1. *(See [18] or [12].) The kernel of the product of the restriction maps*

$$\prod_{i=1}^n \text{res}_{G, E_i} : T(G) \longrightarrow \prod_{i=1}^n T(E_i)$$

is finite.

Our object is to give a new proof that the rank of $T(G)$ is the number n of subgroups in the list E_1, \dots, E_n . To this end, it is only necessary to show that the image of the product of the restriction maps has finite index in $\prod_{i=1}^n T(E_i) \cong \mathbb{Z}^n$ (for E elementary abelian p -group of rank at least 2, we know from Dade's Theorem [14] that $\mathbb{Z} \xrightarrow{\cong} T(E)$ via $m \mapsto \Omega^m k$ for instance). Since obviously $\prod \text{res}_{G, E_i}(\Omega^1 k) = (1, \dots, 1) \in \mathbb{Z}^n$, it is enough for us to prove the following.

Theorem 10.2. *For each $i = 1, \dots, n - 1$, there exists a number d and an endotrivial kG -module $M = M(E_i)$ with the property that $\text{res}_{G, E_i}(M) \cong \Omega^{-d} k$ in $\text{StMod}(kE_i)$ while $\text{res}_{G, E_j}(M) \cong k$ in $\text{StMod}(kE_j)$ for $1 \leq j \leq n$ and $j \neq i$.*

Therefore the classes of the modules $\Omega k, M(E_1), \dots, M(E_{n-1})$ generate a subgroup of finite index in the group of endotrivial modules.

We first need the following result.

Lemma 10.3. *Suppose that G is a finite group that has at least two classes of maximal elementary abelian p -subgroups and has a maximal elementary abelian p -subgroup of rank 2. Let r be the p -rank of G . Let E_1, \dots, E_n be the subgroups of G , defined as above. Then there exists a number d and elements $z, y_1, y_2, x_3, \dots, x_r$ in $H^d(G, k)$ such that the following hold.*

- (1) $\text{res}_{G, Z}(z) \neq 0$ where Z is the center of a Sylow p -subgroup of G .
- (2) $y_1 y_2 = 0$ and moreover $\text{res}_{G, E_1}(y_2) = 0$ and $\text{res}_{G, E}(y_1) = 0$ for E any maximal elementary abelian p -subgroup which is not conjugate to E_1 .

- (3) The set $\{\text{res}_{G,E_1}(z), \text{res}_{G,E_1}(y_1)\}$ is a system of parameters for the ring $H^*(E_1, k)$.
(4) For $j = 2, \dots, n$, the set $\{\text{res}_{G,E_j}(z), \text{res}_{G,E_j}(y_2)\}$ is a system of parameters for the ring $H^*(E_j, k)$.
(5) For any maximal elementary abelian p -subgroup E of rank $s > 2$, the set

$$\{\text{res}_{G,E}(z), \text{res}_{G,E}(y_2), \text{res}_{G,E}(x_3), \dots, \text{res}_{G,E}(x_s)\}$$

is a system of parameters for the ring $H^*(E, k)$.

Proof. Notice first that the hypotheses on G require that the center Z of a Sylow p -subgroup of G be cyclic. It is a straightforward exercise in the application of Quillen's Dimension Theorem [19, 20] (recalled in Section 1) to find elements which satisfy all of the restriction conditions on systems of parameters and on the structure of varieties. The process can be described as follows.

For any elementary abelian p -subgroup E let J_E denote the ideal $\sqrt{\text{Kernel } \text{res}_{G,E}}$. Then J_E is a prime ideal because the ring $H^*(E, k)/(\text{Rad } H^*(E, k))$ is an integral domain. By Quillen's Dimension Theorem, the minimal prime ideals are the ideals J_E where E is a maximal elementary abelian subgroup. The first element, z is chosen so that z is not in J_Z . Now the second element y_1 is chosen to be in the intersection of all J_E for E not conjugate to a subgroup of E_1 , but y_1 not in J_{E_1} . In addition we want the two elements $\text{res}_{G,E_1}(z)$ and $\text{res}_{G,E_1}(y_1)$ to be a system of parameters for $H^*(E_1, k)$ which means that $\text{res}_{G,E_1}(y_1)$ can not be contained in any of the finite number of maximal ideals that contain $\text{res}_{G,E_1}(z)$. We can find such an element y_1 by the following process. Let E be a maximal elementary abelian p -subgroup which is not conjugate to E_1 . Choose a homogeneous element y_E in J_E such that $\text{res}_{G,E_1}(y_E)$ is not contained in any of the maximal ideals that contain $\text{res}_{G,E_1}(z)$. Quillen's Theorem guarantees that we can do this. Now let y_1 be the product of the y_E where E runs through a set of representatives of the conjugacy classes of maximal elementary abelian subgroups that are not conjugate to E_1 . Then, y_1 satisfies the properties (2) and (3) of the lemma.

Now for the element y_2 we can proceed as follows. For each elementary abelian subgroup E of rank 2, which contains Z and is not conjugate to E_1 , choose a homogeneous element u_E with the properties that $u_E \in J_F$ for every elementary abelian subgroup F which has rank 2, contains Z and which is not conjugate to E . Also we want that $\text{res}_{G,E}(u_E)$ is not in any of the finite number of maximal ideals of $H^*(E, k)$ that contain $\text{res}_{G,E}(z)$. This is accomplished exactly as in the previous paragraph. Next, it should be noticed that u_E^m has the same properties. Hence, there is a number d such that for every elementary abelian subgroup E of rank 2, containing Z and not conjugate to E_1 , there is such an element u_E in $H^d(G, k)$ having the properties specified above. So finally, let y_2 be the sum of the elements u_E where E runs through a set of representatives of the conjugacy classes of elementary abelian subgroup of G having rank 2, containing Z and not conjugate to E_1 .

By the construction, the elements y_1 and y_2 satisfy conditions (2), (3) and (4) except possibly for the requirement that $y_1 y_2 = 0$. However, we do know that $\text{res}_{G,E}(y_1 y_2) = 0$ for every maximal elementary abelian subgroup E of G . Hence, by Quillen's Theorem, $y_1 y_2$ is in the Jacobson radical of $H^*(G, k)$ and hence is nilpotent. So there is some m such that

$(y_1 y_2)^m = 0$. We can now check that the elements z , y_1^m and y_2^m satisfy all of the first four conditions of the lemma.

The elements x_3, \dots, x_r can be chosen to satisfy the required condition (5), by similar arguments. We leave this part of the proof to the reader. \square

Proof of Theorem 10.2. Without loss of generality, we can assume that i , in the theorem, is equal to one. We can assume that G satisfies the hypotheses, and hence also the conclusion, of Lemma 10.3. So, let $z, y_1, y_2, x_3, \dots, x_r$ be as in Lemma 10.3.

We are going to construct an endotrivial module X_α as in Definition 7.1, for which we need to define two closed subsets W_1, W_2 of \mathcal{V}_G and an automorphism $\alpha: F(W_1 \cup W_2) \rightarrow F(W_1 \cup W_2)$. We set

$$\begin{aligned} W_1 &= \mathcal{V}_G(z) \cap \text{res}_{G, E_1}^*(\mathcal{V}_{E_1}) \\ W_2 &= \mathcal{V}_G(y_1). \end{aligned}$$

Note that $W_1 \cap W_2 = \emptyset$ because $\text{res}_{G, E_1}(z), \text{res}_{G, E_1}(y_1)$ is a system of parameters in $H^*(E_1, k)$ by Lemma 10.3 (3). Observe also that W_2 contains all components of $\mathcal{V}_G(k)$ except $\text{res}_{G, E_1}^*(\mathcal{V}_{E_1})$, by Lemma 10.3 (2). In addition, the intersection of $W_1 \cup W_2$ with $\text{res}_{G, E_1}^*(\mathcal{V}_{E_1})$ is a finite set of points.

Of particular interest to us is the fact that $W_1 \cup W_2$ contains both closed subsets $\mathcal{V}_G(y_1)$ and $\mathcal{V}_G(z)$. We use this fact to prove the following.

Lemma 10.4. *The element y_1/z is an automorphism of $F(W_1 \cup W_2)$.*

Proof. If $k \xrightarrow{\varepsilon} F$ is a Rickard idempotent then the induced map

$$\underline{\text{End}}_{kG}^*(F) \xrightarrow{\varepsilon^*} \underline{\text{Hom}}_{kG}^*(k, F) = \hat{H}^*(G, F)$$

is an isomorphism. Furthermore, if $k \xrightarrow{\varepsilon} F \xrightarrow{\varepsilon'} F'$ are Rickard idempotents then we have a commutative diagram

$$\begin{array}{ccc} \underline{\text{End}}_{kG}^*(F) & \xrightarrow{\varepsilon'_*} & \underline{\text{Hom}}_{kG}^*(F, F') \xleftarrow{\cong} \underline{\text{End}}_{kG}^*(F') \\ \cong \downarrow \varepsilon^* & & \cong \downarrow \varepsilon^* \swarrow \cong \\ \hat{H}^*(G, F) & \xrightarrow{\varepsilon'_*} & \hat{H}^*(G, F') \end{array}$$

If $x, y \in \underline{\text{End}}_{kG}^*(F)$, let $(\varepsilon')^*(x) = \varepsilon'_*(u)$ and $(\varepsilon')^*(y) = \varepsilon'_*(v)$, i.e., $x\varepsilon' = \varepsilon'u$ and $y\varepsilon' = \varepsilon'v$. Then $(\varepsilon')^*(xy) = xy\varepsilon' = x\varepsilon'v = \varepsilon'uv = \varepsilon'_*(uv)$. It follows that $(\varepsilon')^*(\varepsilon'_*)^{-1}$ is a ring homomorphism.

Apply this to the Rickard idempotents $k \xrightarrow{\varepsilon_2} F(W_2) \xrightarrow{\varepsilon_{21}} F(W_1 \cup W_2)$. We know from Proposition 5.6 that $\underline{\text{End}}_{kG}^*(F(W_2))$ is the localization of $H^*(G, k)$ obtained by inverting y_1 . It follows that y_1 is invertible in $\underline{\text{End}}_{kG}^*(F(W_2))$, and hence applying the argument above, it is invertible in $\underline{\text{End}}_{kG}^*(F(W_1 \cup W_2))$. Likewise, for the cohomology element z , we have homomorphisms $k \rightarrow F(\mathcal{V}_G(z)) \rightarrow F(W_1 \cup W_2)$, and using the same argument we have that z is invertible in $\underline{\text{End}}_{kG}^*(F(W_1 \cup W_2))$. Finally, we recall that the ring of ordinary endomorphisms of $F(W_1 \cup W_2)$ is $\underline{\text{End}}_{kG}^0(F(W_1 \cup W_2))$. \square

Returning to the proof of the theorem, we define the automorphism

$$\alpha := y_1/z \in \text{Aut}(F(W_1 \cup W_2))$$

and we define a module X_α as in Definition 7.1. By Theorem 7.3, X_α is a finite dimensional endotrivial module. We have a weak pullback diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\varepsilon_1^\alpha} & F(W_1) \\ \varepsilon_2^\alpha \downarrow & & \downarrow \alpha \varepsilon_{12} \\ F(W_2) & \xrightarrow{\varepsilon_{21}} & F(W_1 \cup W_2), \end{array} \quad (10.1)$$

as in Remark 7.2. With the notation of Remark 7.6, we have that $X_\alpha = X(\alpha, W_1, W_2)$.

It remains only to identify X_α in terms of its restrictions to elementary abelian subgroups. Let $E \subseteq G$ be a maximal elementary abelian p -subgroup of G . Suppose first that E is not conjugate to E_1 . Then $(\text{res}_{G,E}^*)^{-1}(W_2) = \mathcal{V}_E$ and consequently $(\text{res}_{G,E}^*)^{-1}(W_1) = \emptyset$. So, the restriction of Diagram (10.1) to E is a weak pullback square as in Remark 7.8. Therefore the restriction of X_α to E is isomorphic to the trivial module k .

Suppose on the other hand that $E = E_1$ is conjugate to E_1 . Then by Proposition 8.1, the restriction of X_α to E is isomorphic to $\Sigma^d k = \Omega^{-d} k$. This completes the proof of the theorem. \square

11. GLUING ALONG ENDOMORPHISMS

The reader may note that the construction in the proof of Theorem 10.2 is significantly different from the proofs in the examples of Sections 8 and 9 where the closed sets W_1 and W_2 that are chosen are hypersurfaces. This is still the situation if the group G has p -rank 2. However, if G has larger p -rank, then the set W_1 is a finite union of points, thus having higher codimension. Even here it is possible to prove Theorem 10.2 using hypersurfaces defined by cohomology elements, but to do so we must glue along endomorphisms rather than automorphisms. We end the paper with a brief discussion of how such a gluing can be proved. As this is not essential for the main theorems of the paper, we leave many of the details to the reader. In an earlier version of this paper, Theorem 10.2 was proved using the methods in this section.

Let us recall the general Mayer-Vietoris situation of Sections 4, 6 and 7, that is, we assume that we have an open covering of the projective support variety

$$\mathcal{V}_G = U_1 \cup U_2$$

and we denote by $W_1 = \mathcal{V}_G \setminus U_1$ and $W_2 = \mathcal{V}_G \setminus U_2$ the closed complements.

Our quest, initiated in Remark 7.7, is to find good conditions under which the gluing automorphism $\alpha: k \xrightarrow{\cong} k$ on $U_1 \cap U_2$ could be replaced by a general endomorphism, in our original construction of X_α in Definition 7.1. We already gave a rather trivial answer to this question in Remark 7.8 when $W_1 = \emptyset$.

Definition 11.1. Let $\beta \in \text{End}_{\mathcal{T}}(F(W_1))$ be an endomorphism of $F(W_1)$ in $\mathcal{T} = \text{StMod}(kG)$. Define an object \tilde{X}_β in \mathcal{T} and two morphisms $\tilde{\varepsilon}_1^\beta$ and $\tilde{\varepsilon}_2^\beta$ by the following weak pullback (Definition 3.4):

$$\begin{array}{ccc} \tilde{X}_\beta & \xrightarrow{\tilde{\varepsilon}_1^\beta} & F(W_1) \\ \tilde{\varepsilon}_2^\beta \downarrow & & \downarrow \beta \\ k & \xrightarrow{\varepsilon_1} & F(W_1), \end{array} \quad (11.1)$$

which should be compared to (7.2). Equivalently, we have a distinguished triangle:

$$\tilde{X}_\beta \xrightarrow{\begin{pmatrix} \tilde{\varepsilon}_1^\beta \\ -\tilde{\varepsilon}_2^\beta \end{pmatrix}} F(W_1) \oplus k \xrightarrow{(\beta \ \varepsilon_1)} F(W_1) \rightarrow \Sigma \tilde{X}_\beta. \quad (11.2)$$

Proposition 11.2. Let β be an endomorphism of $F(W_1)$ in \mathcal{T} and assume that it becomes an isomorphism on $U_1 \cap U_2$. Let $\alpha \in \text{Aut}_{\mathcal{T}}(F(W_1 \cup W_2))$ be the restriction of β . Then the object \tilde{X}_β of Definition 11.1 is isomorphic to the endotrivial module X_α of Definition 7.1.

Proof. By Remark 5.7, the hypothesis that the localisation of β is equal to α boils down to the commutativity of the following diagram in \mathcal{T} :

$$\begin{array}{ccc} F(W_1) & \xrightarrow{\beta} & F(W_1) \\ \varepsilon_{12} \downarrow & & \downarrow \varepsilon_{12} \\ F(W_1 \cup W_2) & \xrightarrow[\alpha]{\cong} & F(W_1 \cup W_2). \end{array} \quad (11.3)$$

By Lemma 6.3, the morphism ε_{12} is a U_2 -isomorphism and so is α of course. From this, we deduce by two-out-of-three that β is a U_2 -isomorphism. Therefore, by Proposition 3.3 applied to the distinguished triangle (11.2), the morphism $\tilde{\varepsilon}_2^\beta$ is also a U_2 -isomorphism. By the same proposition for the same triangle, since ε_1 is a U_1 -isomorphism, so is $\tilde{\varepsilon}_1^\beta$. We now have two isomorphisms in $\mathcal{T}(U_1)$ and $\mathcal{T}(U_2)$ respectively, $\sigma_1 := \varepsilon_1^{-1} \circ \tilde{\varepsilon}_1^\beta$ and $\sigma_2 := \tilde{\varepsilon}_2^\beta$.

$$\sigma_1: \tilde{X}_\beta \xrightarrow{\tilde{\varepsilon}_1^\beta} F(W_1) \xleftarrow{\varepsilon_1} k \quad \text{and} \quad \sigma_2: \tilde{X}_\beta \xrightarrow{\tilde{\varepsilon}_2^\beta} k.$$

Computing in $\mathcal{T}(U_1 \cap U_2)$, where all morphisms in sight become isomorphisms, we have

$$\sigma_2 \circ \sigma_1^{-1} \stackrel{\text{def}}{=} \tilde{\varepsilon}_2^\beta \circ (\tilde{\varepsilon}_1^\beta)^{-1} \circ \varepsilon_1 \stackrel{(11.1)}{=} \varepsilon_1^{-1} \circ \beta \circ \varepsilon_1 \stackrel{(11.3)}{=} (\varepsilon_{12} \varepsilon_1)^{-1} \circ \alpha \circ (\varepsilon_{12} \varepsilon_1) =: \sigma.$$

Since $\varepsilon_{12} \varepsilon_1 = \varepsilon_{W_1 \cup W_2}: k \rightarrow F(W_1 \cup W_2)$ by (6.2), the above morphism $\sigma: k \rightarrow k$ is the automorphism of k in $\mathcal{C}(U_1 \cap U_2)$ which corresponds to $\alpha \in \text{Aut}_{\mathcal{T}}(F(W_1 \cup W_2))$, see Corollary 5.4. So, we have proved that the object \tilde{X}_β is isomorphic to k on U_1 and on U_2 , via the isomorphisms σ_1 and σ_2 respectively, and we have $\sigma_2 \circ \sigma_1^{-1} = \sigma$. This means that \tilde{X}_β is the gluing of two copies of k along σ . But we already know from Theorem 7.3 that this gluing is X_α . Therefore, $\tilde{X}_\beta \cong X_\alpha$ by uniqueness of the gluing. \square

We now combine the above modified construction with the trivial Remark 7.8, to get the following statement. The final construction applies even when the p -rank of G is greater than two. For the sake of clarity, we repeat all hypotheses.

Corollary 11.3. *Let G be a finite group and let $W_1, W_2 \subseteq \mathcal{V}_G$ be disjoint closed subsets of its support variety. Let $\gamma: F(W_1 \cup W_2) \rightarrow F(W_1 \cup W_2)$ be an endomorphism in $\mathcal{T} = \mathbf{StMod}(kG)$.*

Define an object $\hat{X}_\gamma \in \mathbf{StMod}(kG)$ and two morphisms $\hat{\varepsilon}_1^\gamma$ and $\hat{\varepsilon}_2^\gamma$ by the following weak pullback (Def. 3.4):

$$\begin{array}{ccc} \hat{X}_\gamma & \xrightarrow{\hat{\varepsilon}_1^\gamma} & F(W_1) \\ \hat{\varepsilon}_2^\gamma \downarrow & & \downarrow \gamma \varepsilon_{12} \\ F(W_2) & \xrightarrow{\varepsilon_{21}} & F(W_1 \cup W_2). \end{array} \quad (11.4)$$

Let $W_3 \subseteq \mathcal{V}_G$ be another closed subset, disjoint from W_1 . Suppose that for every maximal elementary abelian p -subgroup $E \subseteq G$, at least one of the following two conditions holds true:

- (1) $W_1 \cap \text{res}_{G,E}^*(\mathcal{V}_E) = \emptyset$.
- (2) $W_2 \cap \text{res}_{G,E}^*(\mathcal{V}_E) = \emptyset$ and, if we denote by W_i' the preimage of W_i in \mathcal{V}_E via the map $\text{res}_{G,E}^*: \mathcal{V}_E \rightarrow \mathcal{V}_G$ for $i = 1, 3$, the morphism $\text{res}_{G,E}(\gamma): F(W_1') \rightarrow F(W_1')$ is a U -isomorphism (Def. 3.2) where U is the open $\mathcal{V}_E \setminus (W_1' \cup W_3')$ in \mathcal{V}_E .

Then \hat{X}_γ is an endotrivial kG -module.

Its restriction to a subgroup $E \subseteq G$ as above is trivial if E satisfies Condition (1). If E satisfies Condition (2), the restriction $\text{res}_{G,E}(\hat{X}_\gamma)$ is isomorphic to the kE -module X_α obtained from Definition 7.1 for the group E , for the disjoint closed subsets W_1' and W_3' of \mathcal{V}_E and for the automorphism $\alpha := \text{res}_{G,E}(\gamma) \otimes F(W_3') \in \text{Aut}_{\mathbf{StMod}(kE)}(F(W_1' \cup W_3'))$.

Proof. Let us restrict the weak pullback of the statement to a maximal elementary abelian p -subgroup $E \subseteq G$. Assume first that E satisfies Condition (1) then it is clear that $\text{res}_{G,E}(\hat{X}_\gamma) \cong k$ (see Remark 7.8). On the other hand, suppose that E satisfies Condition (2). Note that we then have $\text{res}_{G,E}(F(W_2)) = k$. The restriction to E of the weak pullback (11.4) is isomorphic to

$$\begin{array}{ccc} \text{res}_{G,E}(\hat{X}_\gamma) & \longrightarrow & F(W_1') \\ \downarrow & & \downarrow \text{res}_{G,E}(\gamma) \\ k & \xrightarrow{\varepsilon_1} & F(W_1'). \end{array}$$

So, $\text{res}_{G,E}(\hat{X}_\gamma)$ is a module of the form $\tilde{X}_{\text{res}_{G,E}(\gamma)}$ as in Definition 11.1, applied to the group E , to the endomorphism $\beta = \text{res}_{G,E}(\gamma)$, and to the open covering of \mathcal{V}_E given by the complements of W_1' and W_3' , which are obviously disjoint since $W_1 \cap W_3 = \emptyset$. Proposition 11.2 shows that this kE -module is endotrivial and coincides with the announced module X_α .

We have proved that $\text{res}_{G,E}(\hat{X}_\gamma)$ is endotrivial for all $E \subseteq G$ as above. This is indeed enough by the following folklore result. \square

Proposition 11.4. *Let $M \in \text{StMod}(kG)$ such that the restriction of M to every (maximal) elementary abelian p -subgroup of G is finitely generated (resp. endotrivial), then so is M .*

Proof. Use Chouinard’s Theorem [13] and Frobenius reciprocity to show that the modules induced from elementary abelian subgroups generate the stable module category, see more in [10]. Then use Frobenius reciprocity again to see that an object in the stable category is compact if and only if its restriction to every elementary abelian subgroup is compact. \square

Remark 11.5. To prove Theorem 10.2 using this Corollary 11.3, we set $W_1 = \mathcal{V}_G(z) \cap \text{res}_{G,E_1}^*(\mathcal{V}_{E_1})$, $W_2 = \mathcal{V}_G(z) \cap \bigcup_{E \in \mathcal{E}} \text{res}_{G,E}^*(\mathcal{V}_E)$, where \mathcal{E} is the family of maximal elementary abelian p -subgroups E which are not conjugate to E_1 , and finally we set $W_3 = \mathcal{V}_G(y_1)$. Note that $W_2 \subseteq W_3$ and that W_1 and W_3 are called W_1 and W_2 in the proof of 10.2. The endomorphism of $F(W_1 \cup W_2)$ that we use is y_1/z which becomes an automorphism when restricted to $F(W_1 \cup W_3)$, as we have observed.

REFERENCES

- [1] J. L. Alperin, *A construction of endo-permutation modules*, J. Group Theory **4** (2001), 3–10.
- [2] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine & Angew. Math. **588** (2005), 149–168.
- [3] P. Balmer and G. Favi, *Gluing techniques in triangular geometry*, Q. J. Math. **58** (2007), 415–441.
- [4] D. J. Benson, *Representations and Cohomology II: Cohomology of groups and modules*, Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, 1991 (paperback reprint 1998).
- [5] D. J. Benson, J. F. Carlson, and J. Rickard, *Complexity and varieties for infinitely generated modules, I*, Math. Proc. Camb. Phil. Soc. **118** (1995), 223–243.
- [6] ———, *Complexity and varieties for infinitely generated modules, II*, Math. Proc. Camb. Phil. Soc. **120** (1996), 597–615.
- [7] D. J. Benson and G. Ph. Gnacadja, *Phantom maps and purity in modular representation theory, I*, Fundamenta Mathematicae **161** (1999), 37–91.
- [8] J. F. Carlson, *The variety of an indecomposable module is connected*, Invent. Math., **77** (1984), 291–299.
- [9] ———, *Modules and Group Algebras*, Lectures in Mathematics, ETH Zürich, Birkhäuser, 1996.
- [10] ———, *Cohomology and induction from elementary abelian subgroups*, Q. Math. J. **51** (2000), 169–181.
- [11] J. F. Carlson, *Constructing endotrivial modules*, J. Pure and Appl. Algebra, **206**(2006), 83–110.
- [12] J. F. Carlson and J. Thévenaz, *The classification of endo-trivial modules*, Invent. Math. **158** (2004), 389–411.
- [13] L. Chouinard, *Projectivity and relative projectivity over group rings*, J. Pure & Applied Algebra **7** (1976), 278–302.
- [14] E. C. Dade, *Endo-permutation modules over p -groups, II*, Ann. of Math. **108** (1978), 317–346.
- [15] E. Friedlander and J. Pevtsova, *Π -supports for modules for finite group schemes*, Duke Math. J. **139** (2007), 317–368.
- [16] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, Berlin/New York, 1977.
- [17] M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc., vol. 128, American Math. Society, 1997.
- [18] L. Puig, *Affirmative answer to a question of Feit*, J. Algebra **131** (1990), 513–526.
- [19] D. G. Quillen, *The spectrum of an equivariant cohomology ring, I*, Ann. of Math. **94** (1971), 549–572.
- [20] ———, *The spectrum of an equivariant cohomology ring, II*, Ann. of Math. **94** (1971), 573–602.
- [21] J. Rickard, *Idempotent modules in the stable category*, J. London Math. Soc. **178** (1997), 149–170.

- [22] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque, vol. 239, Société Math. de France, 1996.
- [23] C. A. Weibel, *Introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, 1994.

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