Facts about series

1. General

Definition. A series $a_1+a_2+\cdots$ *converges* with sum S, written symbolically as $a_1+a_2+\cdots=S$ or $\sum_{n=1}^{\infty} a_n = S$, when its sequence of partial sums converges to S.

Observation. If you change a finite number of terms of a series, or put more terms in, or leave some terms off (finitely many), then you don't affect whether the new series converges, but if it does converge the new sum will likely be different.

(In contrast, with convergent sequences, if you make a finite number of changes there is no effect on the limit.)

2. Series of positive terms

We might as well say "positive terms" rather than "nonnegative terms", because if some terms are zero we can always ignore them.

Theorem *** (Boundedness test) A series of positive terms converges $\Leftrightarrow$ its partial sums are bounded.

Theorem *** (Comparison test) If $0 \le a_n \le b_n$ for each $n = 1, 2, \cdots$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Put the other way around, if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n abn$ diverges.

Theorem *** (Integral test) If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and $a_n = f(n)$ for some decreasing continuous function $f(x)$, then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \int_1^{\infty}$ converges.

Example: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. It converges for $p > 1$. The integral test applies when $p > 0$ (so that f is decreasing). (The p-series is still divergent for $p \le 0$ even though the integral test doesn't apply.)

Theorem *** (Comparison by dividing) (1) If a_n and b_n are series of positive terms and $\frac{a_n}{b_n} \to c \ne 0$ as $n \to \infty$, then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or else they both diverge.

(2) If instead $\frac{a_n}{b_n} \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(In (2), the idea is that $\sum_{n=1}^{\infty} a_n$ must eventually be a lot smaller than $\sum_{n=1}^{\infty} b_n$.)

Theorem *** (Ratio test) If $\sum_{n=1}^{\infty} a_n$ $(n = 1, 2, \cdots)$ is a series of positive terms and $\frac{a_{n+1}}{a_n} \to r$ as $n \to \infty$, then

$$\begin{cases} \text{if } r < 1 & \text{the series converges} \\ \text{if } r > 1 & \text{the series diverges} \\ \text{if } r = 1 & \text{you can't tell} \end{cases}$$

(The idea is that if $r \neq 1$ then $\sum_{n=1}^{\infty} a_n$ looks pretty much like a geometric series. If $r = 1$ then the situation is too delicate to tell.)

(Don't confuse the ratio test with comparison by dividing; both have ratios but they're ratios of different things.)

(The ratio test appears in the section on series with positive and negative terms, but with absolute values, so the idea is really the same as here.)

3. Series of positive and negative terms

In other words, this section is about *any* series.

We say that a series $\sum_{n=1}^{\infty} a_n$ **is absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem *** An absolutely convergent series is convergent.

If a series $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then we say that this series is *conditionally convergent.* An odd fact:

Theorem *** By re-ordering the terms of a conditionally convergent series, you can make the new series converge to any sum you wish.

An **alternating series** is a series where the terms are alternately positive and negative.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, the *alternating harmonic series.*

Theorem *** An alternating series whose terms are decreasing in absolute value converges. Further, the error at any term (the distance of the corresponding partial sum from the sum of the whole series) is less in absolute value than the following term.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$, and the error after n terms is at most $\frac{1}{n+1}$. (This is not very good; in other words, the series doesn't converge very fast.)