Concepts for algebras

1. Definitions

- A *term* t or $t(x_1, \ldots, x_n)$ of type τ is a formal expression as a string of symbols, defined recursively as follows, starting from variable symbols $x_1, \ldots, x_n$ for τ:

  (a) Each x_i is a term, and

  (b) if $t_1, \ldots, t_{n_\gamma}$ are terms, so is $\mathbf{f}_\gamma(t_1, \ldots, t_{n_\gamma})$, where $\mathbf{f}_\gamma$ and the commas and parentheses are symbols and $\gamma \in \Gamma$.

- For elements $a_1, \ldots, a_n$ of an algebra A, the *value* $t(a_1, \ldots, a_n)$ is the element of A obtained by using $t(x_1, \ldots, x_n)$ as a recipe.

- A *term relation* $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$ is an equation holding for a *particular* n-tuple of elements of A.

- A *law* is a formal equation $t_1 = t_2$ or $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$, with $(\forall x_1) \ldots (\forall x_n)$ understood. The law $t_1 = t_2$ *holds* in A when *all* n-tuples $a_1, \ldots, a_n$ from A satisfy the term relation $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$.

  We also say A satisfies $t_1 = t_2$, or A is a model of $t_1 = t_2$, or write $A \models t_1 = t_2$.

- A *variety* of algebras of a given type is the class of all models of some set of laws. Examples are the varieties of all groups, of all abelian groups, of all lattices, and of all distributive lattices.

  If A is an algebra we write $\mathrm{Var}(A)$ for the variety determined by all laws holding in A, which is the smallest variety containing A.

- A *subalgebra* of an algebra A is a subset $S \subseteq A$ that is closed under all the basic operations of A.

- An algebra A is said to be *generated* by its elements $g_1, \ldots, g_n$ if the smallest subalgebra of A that contains all the g_i is A itself.

- A *homomorphism* $\phi : A \to B$ between similar algebras is a map compatible with the basic operations of A and B.

- The *direct product* of a family of similar algebras, $A_1 \times A_2$ or more generally $\prod_{\gamma \in \Gamma} A_\gamma$, is the set-theoretic cartesian product with operations computed coordinatewise.

- A *congruence relation* on A is an equivalence relation θ on A that is compatible with the basic operations of A.

- For a congruence relation θ on A, the blocks of θ form an algebra A/θ of the same type, with a natural surjective homomorphism $\eta : A \to A/\theta$.

2. Some theorems

Familiar theorems from group theory all generalize, except that in groups we focus on normal subgroups, but for algebras in general we focus on congruence relations, a generalization of the coset decomposition of a normal subgroup. The reason is that in groups the whole coset decomposition is determined by knowing the block containing the identity element, while for algebras in general no one block determines the rest.

- The subalgebra of A generated by $g_1, \ldots, g_n$ is the set of elements of the form $t(g_1, \ldots, g_n)$ for some term t in n variables.

- The image of a homomorphism is a subalgebra.

- The set $\mathrm{Con}(A)$ of all congruence relations on A is a lattice, the *congruence lattice* of A.

- If $\phi : A \to B$ is a homomorphism, then the equivalence relation on A induced by ϕ is a congruence relation, which we call $\ker \phi$, the *kernel* of ϕ.

  Observe that if $\theta \in \mathrm{Con}(A)$ and $\eta : A \to A/\theta$ is the natural surjection, then $\ker \eta = \theta$.

- If $\phi : A \to B$ is a surjective homomorphism, then $B \cong A/\ker \phi$ (the **first isomorphism theorem**).

  Thus we have an "internal description" of all the homomorphic images of A, up to isomorphism.

- If $\phi : A \to B$ is a surjective homomorphism, then the congruence relations on B correspond one-to-one to the congruence relations on A that contain $\ker \phi$ (the **correspondence theorem**).

- For a direct product $P = \prod_{\gamma \in \Gamma} A_\gamma$, for each $\gamma \in \Gamma$ the coordinate projection $\pi_\gamma : P \to A_\gamma$ is a surjective homomorphism.