Algebras

We use the term "algebra" to mean an algebraic system—a set with operations.

1. Examples

(1) A group $\langle G; \cdot, {}^{-1}, e\rangle$.

(2) A ring $\langle R; +, \cdot, -, 0\rangle$; or a ring with 1 $\langle R; +, \cdot, -, 0, 1\rangle$.

(3) A Boolean algebra $\langle \mathrm{B}; \vee, \wedge, 0, 1, {}'\rangle$.

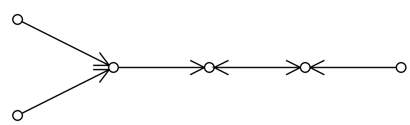
(4) A lattice $\langle L; \vee, \wedge\rangle$; the lattice $\langle \mathbf{R}; \max, \min\rangle$.

(5) A vector space $\langle V; +, -, 0, \text{mult by } r \text{ for each } r \in \mathbf{R}\rangle$ (if V is over the reals).

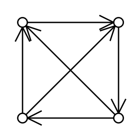
(6) Perkins' semigroup $\langle S; \cdot\rangle$, with elements

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(7) The 1-unary algebra $\langle A; f\rangle$ with diagram



(8) The tournament $\langle T; \vee, \wedge\rangle$ with diagram



(9) The Heyting algebra $\langle \{0, a, 1\}; \vee, \wedge, \rightarrow, 0, 1\rangle$.

(10) The Murskii 1-binary algebra $\langle M; \cdot\rangle$ with table

	0	a	b
0	0	0	0
a	0	0	a
b	0	b	b

(11) Tarski's high-school-algebra algebra $\langle \omega; +, \cdot, \uparrow, 1\rangle$.

(12) Shallon's graph algebra $\langle G \cup \{0\}; \cdot\rangle$, $\quad G =$

(13) The relation algebra $\langle \mathrm{Pow}(S \times S); \cup, \cap, \emptyset, 1, ', \circ, {}^{\cup}, \Delta \rangle$ (S any set).

(14) The implication algebra $\langle \mathbf{2}; \rightarrow \rangle$.

(15) The lattice-ordered group $\langle \mathbf{Z}; \wedge, \vee, +, -, 0 \rangle$.

(16) The set algebra $\langle S; \rangle$ (set S with no operations).

(17) The 1-binary algebra $\langle \{0, 1, 2\}; \cdot \rangle$ with table

	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

2. Some sets of laws

[1] Defining laws for groups: A *group* is an algebra ... satisfying the laws ...

[2] Defining laws for lattices: A *lattice* is an algebra ... satisfying the laws ...

[3] Defining laws for relation algebras: A *relation algebra* is an algebra $\langle R; \vee, \wedge, 0, 1, ', \circ, {}^{\cup}, \Delta \rangle$ such that

(i) $\langle R; \vee, \wedge, 0, 1 \rangle$ is a Boolean algebra;

(ii) $\circ$ is associative;

(iii) $\Delta \circ x = x \circ \Delta = x$;

(iv) ${}^{\cup}$ is a Boolean automorphism, $x^{\cup\cup} = x$, and $(x \circ y)^{\cup} = y^{\cup} \circ x^{\cup}$;

(v) $(x \circ y) \wedge z \leq x \circ (y \wedge (x^{\cup} \circ z))$.

[4] Defining laws for Heyting algebras: A *Heyting algebra* is an algebra $\langle H; \vee, \wedge, \rightarrow, 0 \rangle$ such that

(i) $\langle H; \vee, \wedge, 0 \rangle$ is a lattice with 0;

(ii) $x \wedge (x \rightarrow y) = x \wedge y$;

(iii) $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$;

(iv) $z \wedge ((x \wedge y) \rightarrow x) = z$.

[5] Defining laws for implication algebras: An *implication algebra* is an algebra $\langle A; \rightarrow \rangle$ such that

(i) $(x \rightarrow y) \rightarrow x = x$;

(ii) $(x \to y) \to y = (y \to x) \to x$;

(iii) $x \to (y \to z) = y \to (x \to z)$.

[6] Tarski's "high-school identity problem": Do these laws imply all laws of $\langle \omega; x+y, xy, x^y, 1\rangle$? This was solved; the answer is negative.

$$\begin{array}{llll} x+y=y+x & xy=yx & x+(y+z)=(x+y)+z & x(yz)=(xy)z \\ x(y+z)=xy+xz & x^{y+z}=x^y x^z & (xy)^z = x^z y^z & (x^y)^z = x^{(yz)} \\ x\cdot 1 = x & x^1 = x & 1^x = 1 & \end{array}$$

[7] Robbins' Problem: Do these laws define Boolean algebras? The answer is "yes"; the proof was found by computer in 1996.

(i) $\vee$ is commutative;

(ii) $\vee$ is associative;

(iii) $((x \vee y)' \vee (x \vee y')')' = x$.

Erratum: A different version of (iii) was quoted in Handout H; it was not the one Robbins asked about.

3. Some definitions

3.1 A function $f : A^n \to A$ is an *n-ary operation* on A; n is its "arity."

(For $n = 0, 1, 2, 3$ we say "nullary", "unary", "binary", "ternary".)

3.2 An *algebra* is a set A with a given family of operations f_γ $(\gamma \in \Gamma)$, called the "basic operations" of A. Officially, the algebra is $\langle A; f_\gamma, \gamma \in \Gamma\rangle$. Texts often use a separate letter to distinguish the algebra from the set, but we'll follow the informal practice of group theory and use A for both.

3.3 The *type* of $\langle A; f_\gamma, \gamma \in \Gamma\rangle$ is the function $\tau : \Gamma \to \omega$ given by $\tau(\gamma) = n_\gamma$, the arity of f_γ. Two algebras of the same type are *similar*. In discussions involving more than one algebra, we'll normally assume that all the algebras are similar. Usually Γ will be finite; if $|\Gamma| = m$, then it is simplest to choose $\Gamma = 0, \ldots, m-1$ and write the n_γ as a sequence.

For example, the type of a Boolean algebra $\langle B; \vee, \wedge, 0, 1, '\rangle$ can be written $\langle 2, 2, 0, 0, 1\rangle$.

3.4 The *terms* (formal expressions) in variable symbols $x_1, \ldots, x_n$ for type τ are the strings of symbols obtained recursively from these conditions:

(a) Each x_i is a term, and

(b) if $t_1, \ldots, t_{n_\gamma}$ are terms, so is $\mathbf{f}_\gamma(t_1, \ldots, t_{n_\gamma})$, where $\mathbf{f}_\gamma$ and the commas and parentheses are symbols and $\gamma \in \Gamma$.

A term t in variable symbols $x_1, \ldots, x_n$ is often described by $t(x_1, \ldots, x_n)$. For algebras with familiar notations we use those notations instead; for example, a group term might be written as $x_1(x_2^{-1}x_3)$.

3.5 Evaluation: If $t(x_1, \ldots, x_n)$ is a term for type τ and A is an algebra of type τ, and if $a_1, \ldots, a_n \in A$ are given, $t(x_1, \ldots, x_n)$ can be regarded as a recipe for calculating a *value* in A, called $t(a_1, \ldots, a_n)$. Thus t induces a function on $A^n \to A$. The functions so induced are called the *n-ary term functions* on A.

This is similar to the case of of polynomials over a commutative ring R, where we distinguish between a formal polynomial $f(X)$ and a polynomial function. Indeed, if R is finite, there are only finitely many one-variable polynomials functions on R, but $R[X]$ is infinite.

3.6 A sentence $(\forall x_1) \ldots (\forall x_n) t(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)$ is called a *law* or *identity* in n variables. We often suppress $\forall x_i$ or even write $t = u$. Also, many authors write $t \approx u$, to distinguish this formal situation from actual equality of two elements. If $t(a_1, \ldots, a_n) = u(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$, then $t = u$ is *satisfied* by A (written $A \models t = u$), or *holds* in A, or that A is a *model* of $t = u$.

3.7 A *variety* is a class of algebras definable by laws, i.e., the class of all algebras that satisfy some particular set of laws. An *equational theory* is the set of all laws satisfied by some one class of similar algebras.

Example of varieties are those of rings, of groups, of abelian groups, of lattices, of distributive lattices, and of the other classes of algebras whose defining laws are given in §2.

4. Problems

Problem A-1. For each of these algebras K, find (i) a 1-variable law of the algebra that does not hold in *all* algebras of the same type, and (ii) (if you can) a law in 2 or more variables that is not an obvious consequence of a 1-variable law of the algebra. No proofs are required.

(a) Perkins' semigroup;

(b) Murskii's 1-binary algebra;

(c) Shallon's graph algebra [note: the operation is idempotent];

(d) the permutation group S_3.

(e) the tournament (8).

(A *tournament* is a directed graph in which every two vertices are joined by a single edge oriented one way or the other. It can be envisioned as a record of who won each match in a "round-robin" tournament, where each player has played every other player once—the arrow points towards the player who won. A tournament can be made into an algebra by letting $x \vee y$ be the winner and $x \wedge y$ the loser of the game between x and y.)

Problem A-2. For the 1-unary algebra $\langle A; f \rangle$ of Example (7), find its equational theory (the set of all laws that hold). You'll need to consider the possibilities $f^n(x) = f^m(y)$ and $f^n(x) = f^m(x)$ $(m \geq n \geq 0)$. Sketch your reasoning.

Problem A-3. For the two-element group $C_2 = \{e, a\}$, invent a procedure for telling whether a given group law holds in C_2. (For example, $((xy)z^{-1})^{-1} = x^{-1}(zy)$?)

Problem A-4. For each of the algebras of examples (4)(for $\mathbf{R}$), (6), (7), (8), (10), (12), (15), (16), (17) in §1, comment on its subalgebras. If there are just a couple, say what they are; if there are many, either describe them all or describe a typical one. No proofs are required.

Problem A-5. Of the binary operations involved in the examples from §1, list those that are *not* commutative.