Algebraic lattices

1. The concept

In a complete lattice L, we say that an element $c \in L$ is *compact* if $c \leq \sup S$ for $S \subseteq L$ implies $c \leq \sup S_0$ for some finite subset S_0 of S.

We say that a complete lattice L is *algebraic* if every element of L is the sup of a set of compact elements. Examples are

(a) Subgroups(G) for a group G, where the compact elements are the finitely generated subgroups;

(b) Subspaces(V) for a vector space V, where the compact elements are the finite-dimensional subspaces; and

(c) Ideals(R) for a ring R, where the compact elements are the finitely generated ideals.

(d) Ideals(M) for a lattice M, where the compact elements are the principal ideals. (Observe that finitely generated ideals are principal.)

(e) Con(A) for an algebra A, where the compact elements are the principal congruence relations and their finite joins.

2. Completely meet-irreducible elements

In a complete lattice L, we say that an element m is *completely meet-irreducible* (c.m.i.) if $m = \inf S$ implies $m \in S$, for any $S \subseteq L$.

Observe that c.m.i. elements are m.i., and conversely in a finite lattice. For m.i. elements m we required $m < 1$; for c.m.i. elements this is automatic since in the definition we could take S to be empty.

Also observe that each c.m.i. element m has a unique covering element m^+, where $m^+ = \inf\{x : x > m\}$. In fact, the existence of such a covering element is equivalent to being c.m.i.

An example of a m.i. element that is not c.m.i. is any member of the chain $\mathbf{R}$ of reals.

Theorem. In an algebraic lattice L, every element is the inf of a set of c.m.i. elements.

Outline of proof. For any element x and compact element $c \not\leq x$, a Zorn argument gives an element m maximal with respect to the properties $x \leq m, c \not\leq m$. Such an m is c.m.i. Therefore the inf of c.m.i. elements $> x$ has no compact elements under it except those under x, so equals x since every element is the sup of a set of compact elements.

Corollary. In a distributive lattice, every ideal is an intersection of prime ideals (possibly infinitely many).

Proof. Ideals(D) is algebraic, so every element is the inf of a set of c.m.i. elements. Ideals(D) is also easily shown to be distributive, so its c.m.i. elements are meet-prime. But a meet-prime element of the ideal lattice is a prime ideal.

3. Problems

Problem ZZ-1. Justify the examples of §1 in more detail.

Problem ZZ-2. (as in Burris and Sankappanavar) In an algebraic lattice L, let C be the subset of compact elements. (a) Show that C is a join-sub-semilattice of L. (b) Show that $L \cong \text{Ideals}(C)$, where the definition of an ideal in a join-semilattice is the same as for an ideal in a lattice.