Jónsson's Lemma

1. A finite version

Theorem. (Foster) Let A be a finite algebra such that $\mathrm{Var}(A)$ is congruence-distributive. Let $B \in \mathrm{Var}(A)$ be finite and subdirectly irreducible. Then $B \in \mathbf{HS}(A)$.

Corollary. Under the same hypotheses, $|B| \leq |A|$, and if $|B| = |A|$ then $B \cong A$.

Example. Each of the lattices M_3, N_5 satisfies a law that fails in the other.

Proof of the theorem: $\mathrm{Var}(A) = \mathbf{HSP}(A)$, so represent B as a homomorphic image of a subalgebra C of $A \times \cdots \times A$: $C \subseteq A \times \cdots \times A$ and $\phi : C \to B$ (a surjection). Here we know a finite product will do since B is the image of a free algebra $\mathrm{Var}_A(n)$, where $n = |B|$, and such a free algebra can be constructed by the table method. See the left-hand side of Figure 1.

$$\begin{array}{ccccccc} C \subseteq & A \times \cdots \times & A & & & S_{i_0} \subseteq A \\ \downarrow \phi & \swarrow \pi_1 & \searrow \pi_n & & \text{gives} & \downarrow \psi \\ B & A & A & & & B \end{array}$$

Figure 1: Mappings for Foster's Theorem

Focus on $\mathrm{Con}(C)$. One of its elements is $\ker\phi$, which by the Correspondence Theorem is meet-irreducible. Some other elements are the kernels of the coordinate projections restricted to C: $\ker(\pi_i|_C)$. Of course $\pi_i|_C$ may not map C onto A; its image is some subalgebra S_i of A.

Observe that

$$\cap_i \ker(\pi_i|_C) = 0 \leq \ker\phi.$$

Recall that in a distributive lattice, a meet-irreducible element is meet-prime. Therefore $\ker(\pi_{i_0}|_C) \leq \ker\phi$ for some i_0. This says that $\pi_{i_0}(a) = \pi_{i_0}(a') \Rightarrow \phi(a) = \phi(a')$. Therefore a well defined map ψ of the image of S_{i_0} onto B is obtained by setting $\psi(\pi_{i_0}(a)) = \phi(a)$. This map is the desired homomorphism showing that $B \in \mathbf{HS}(A)$. See the right-hand side of Figure 1. □

2. Ultrafilters

A *filter* is the same thing as a dual ideal in the lattice of all subsets of a set—a power-set Boolean lattice. An *ultrafilter* is a maximal dual ideal in such a lattice.

Recall that in a Boolean algebra, choosing one prime ideal gets us a little burst of terminology: the prime ideal is also a maximal ideal, and its complement is a maximal dual ideal (an ultrafilter, if in a power-set lattice).

Now let $I = \omega = \{0, 1, 2, \dots\}$ and let $B = \mathrm{Pow}(I)$. Recall that there are two kinds of prime ideals in B:

- Principal prime ideals. Each is generated by $I \setminus \{k\}$ for some k.

  The complement of such a prime ideal is the "principal ultrafilter" consisting of all subsets containing $\{k\}$.

- Non-principal prime ideals. They exist, since the ideal I_0 of all finite subsets of I, like any ideal in a distributive lattice, is the intersection of prime ideals, and none of them fit the description of a principal prime ideal. Conversely, any non-principal prime ideal must contain I_0; otherwise it would omit some k and be contained in a principal prime ideal, and so, being maximal, would equal that ideal.

  There are $2^{2^{\aleph_0}}$ non-principal prime ideals, if we use the Axiom of Choice, but it is impossible to give even one explicitly!

We shall often treat I as an index set.

Choose a prime ideal in $\mathrm{Pow}(I)$ and keep it fixed for the rest of this discussion. We think of its members as "small" sets of indices. What is a "large" set of indices? There are two possible definitions:

(1) A large set of indices is a set of indices that is not small—a member of the corresponding ultrafilter;

(2) a large set of indices is the complement in I of a small set of indices.

But these two definitions are equivalent! Recall that for a prime ideal in a Boolean lattice, for each x exactly one of x or x' is in the ideal.

Question. For the principal prime ideal generated by $I \setminus \{k\}$, which subsets of I are small and which large? (It is as if only k counts for largeness.)

To summarize, if we have chosen a prime ideal, then with respect to it,

1. Every subset of I is either large or small (not both).

2. The complement in I of a large set is small and vice-versa.

3. The small sets form our chosen prime ideal, by definition. In particular,

   - the union of two small subsets is small;
   - a subset of a small subset is small.
   - The empty set is small.

4. If the prime ideal is nonprincipal, then any finite subset of I is small.

5. The large sets form an ultrafilter. In particular,

   - the intersection of two large subsets is large;
   - a superset of a large subset is large.
   - I itself is large.

3. Ultraproducts

An "ultraproduct" of algebras is their direct product modulo a congruence relation constructed from a nonprincipal ultrafilter. The congruence relation tends to collapse the product down to something that looks like a "generic" copy of the individual algebras, reflecting whatever features they have in common.

The construction is set-theoretic and actually works for sets with relations as well as for algebras. In detail:

Definition. Let I be an infinite index set. Let algebras $A_i, i \in I$ be given. Choose a nonprincipal ultrafilter $\mathcal{U}$ on I. On the direct product $\prod_{i \in I} A_i$, define an relation $\equiv$ by saying $\mathbf{a} \equiv \mathbf{b}$ when $\mathbf{a}$ and $\mathbf{b}$ agree on a large set of indices. The *ultraproduct* of the A_i is the direct product modulo $\equiv$:

$$A^* = (\textstyle\prod_{i \in I} A_i)/\equiv, \text{ or more simply } A^* = \prod_{i \in I} A_i/\mathcal{U}.$$

There are several things to consider here:

- Does the phrase "agree on a large set of indices" mean that there is *some* large set $J \subseteq I$ of indices such that $a_j = b_j$ for all $j \in J$, or that the set of *all* $i \in I$ with $a_i = b_i$ is large? By the properties of large sets, it doesn't matter; the meanings are the same.

- It must be checked that $\equiv$ is an equivalence relation. This follows from the properties of large sets.

- It must also be checked that $\equiv$ is a congruence relation, so that the ultraproduct is an algebra.
- We say "the" ultraproduct even though the result does depend on the choice of $\mathcal{U}$.

Ultraproducts have some startling properties:

1. Any n-ary relation common to the A_i has a reasonable definition on their ultraproduct.
2. Any first-order sentence true in the A_i is true in their ultraproduct. (This extends to first-order formulas.)
3. An ultraproduct of fields is a field. (Why?)
4. The ultraproduct is unchanged (up to isomorphism) if finitely many factors are omitted. (Why?)
5. If all the A_i are finite and isomorphic, then A^* is a copy of the same algebra. (Why?)

Examples.

(a) The ultraproduct of countably many copies of the field $\mathbf{R}$ of reals is the field $\mathbf{R}^*$ of "nonstandard reals". It is possible to do calculus using "infinitesimals" in $\mathbf{R}^*$.

(b) The ultraproduct of countably many copies of the ring $\mathbf{Z}$ of integers is the ring $\mathbf{Z}^*$ of "nonstandard integers". Some of them are "infinite".

(c) The ultraproduct $\mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_5 \times \cdots / \mathcal{U}$ is a field of characteristic 0.

(d) The ultraproduct of chains $\mathbf{1} \times \mathbf{2} \times \mathbf{3} \times \cdots / \mathcal{U}$ is an infinite chain. (What does it look like?)

4. Jónsson's Lemma

"Jónsson's Lemma" would be called a theorem by most people, but it was called a lemma in the original paper and the name has stuck.

For a class $\mathcal{K}$ of similar algebras, let $\mathbf{U}(\mathcal{K})$ denote the class of algebras isomorphic[1] to ultraproducts of algebras in $\mathcal{K}$.

Theorem. (Jónsson's Lemma) Let $\mathcal{K}$ be a class of similar algebras such that $\mathrm{Var}(\mathcal{K})$ is congruence-distributive. If $B \in \mathrm{Var}(\mathcal{K})$ is subdirectly irreducible, then $B \in \mathbf{HSU}(\mathcal{K})$.

[1]Most authors write $\mathbf{P}_U$, following Jónsson, and some omit the use of isomorphic copies.

This theorem doesn't sound much different from the theorem that $\mathrm{Var}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$, but it is really substantially different, in that $\mathbf{U}$ preserves many more properties than $\mathbf{P}$.

Corollary. For a finite algebra A, if $\mathrm{Var}(A)$ is congruence-distributive, then for each subdirectly irreducible algebra $B \in \mathrm{Var}(A)$ we have $B \in \mathbf{HS}(A)$.

Notice that this Corollary is a little stronger than the Theorem of §1, since it is not assumed to start with that B is finite. The conclusion is the same.

5. Problems

Problem DD-1. How can we be sure that an ultraproduct of chains is a chain?

Problem DD-2. Prove the Corollary of §4 from Jónsson's Lemma.

Problem DD-3. Let $\mathbf{F}_4$ be the Galois field of 4 elements, as a ring. Find all the SI members of $\mathrm{Var}(\mathbf{F}_4)$, up to isomorphism. (You may use the fact that $\mathbf{F}_4$ is congruence-distributive.)

Problem DD-4. True or false? "Every lattice satisfies the same laws as its dual." If true, give a brief proof; if false, give a lattice that is a counterexample, with brief explanation. (Either way, it is not necessary to give any specific laws.)

Problem DD-5. Let $\mathcal{K}$ be the class of all lattices of width at most 5. Show that each subdirectly irreducible member of $\mathrm{Var}(\mathcal{K})$ is in $\mathcal{K}$.