Algebras

We use the term "algebra" to mean an algebraic system—a set with operations.

1. Examples

(1) A group $\langle G; \cdot, {}^{-1}, e\rangle$.

(2) A ring $\langle R; +, \cdot, -, 0\rangle$; or a ring with 1 $\langle R; +, \cdot, -, 0, 1\rangle$.

(3) A Boolean algebra $\langle \mathrm{B}; \vee, \wedge, 0, 1, ' \rangle$.

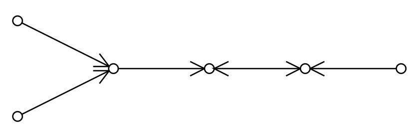
(4) A lattice $\langle L; \vee, \wedge\rangle$; the lattice $\langle \mathbf{R}; \max, \min\rangle$.

(5) A vector space $\langle V; +, -, 0, \text{mult by } r \text{ for each } r \in \mathbf{R}\rangle$ (if V is over the reals).

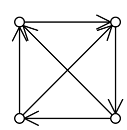
(6) Perkins' semigroup $\langle S; \cdot\rangle$, with elements

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(7) The 1-unary algebra $\langle A; f\rangle$ with diagram



(8) The tournament $\langle T; \vee, \wedge\rangle$ with diagram



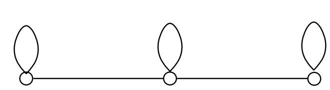
(9) The Heyting algebra $\langle \{0, a, 1\}; \vee, \wedge, \rightarrow, 0, 1\rangle$.

(10) The Murskii 1-binary algebra $\langle M; \cdot\rangle$ with table

	0	a	b
0	0	0	0
a	0	0	a
b	0	b	b

(11) Tarski's high-school-algebra algebra $\langle \omega; +, \cdot, \uparrow, 1\rangle$.

(12) Shallon's graph algebra $\langle G \cup \{0\}; \cdot\rangle$, $G =$



(13) The relation algebra $\langle \mathrm{Pow}(S \times S); \cup, \cap, \emptyset, 1,', \circ, {}^{\cup}, \Delta \rangle$ (S any set).

(14) The implication algebra $\langle \mathbf{2}; \to \rangle$.

(15) The lattice-ordered group $\langle \mathbf{Z}; \wedge, \vee, +, -, 0 \rangle$.

(16) The set algebra $\langle S; \rangle$ (set S with no operations).

(17) The 1-binary algebra $\langle \{0, 1, 2\}; \cdot \rangle$ with table

	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

2. Some sets of laws

[1] Defining laws for groups: A *group* is an algebra ... satisfying the laws ...

[2] Defining laws for lattices: A *lattice* is an algebra ... satisfying the laws ...

[3] Defining laws for relation algebras: A *relation algebra* is an algebra $\langle R; \vee, \wedge, 0, 1,', \circ, {}^{\cup}, \Delta \rangle$ such that

(i) $\langle R; \vee, \wedge, 0, 1 \rangle$ is a Boolean algebra;

(ii) $\circ$ is associative;

(iii) $\Delta \circ x = x \circ \Delta = x$;

(iv) ${}^{\cup}$ is a Boolean automorphism, $x^{\cup\cup} = x$, and $(x \circ y)^{\cup} = y^{\cup} \circ x^{\cup}$;

(v) $(x \circ y) \wedge z \leq x \circ (y \wedge (x^{\cup} \circ z))$.

[4] Defining laws for Heyting algebras: A *Heyting algebra* is an algebra $\langle H; \vee, \wedge, \to, 0 \rangle$ such that

(i) $\langle H; \vee, \wedge, 0 \rangle$ is a lattice with 0;

(ii) $x \wedge (x \to y) = x \wedge y$;

(iii) $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$;

(iv) $z \wedge ((x \wedge y) \to x) = z$.

[5] Defining laws for implication algebras: An *implication algebra* is an algebra $\langle A; \to \rangle$ such that

(i) $(x \to y) \to x = x$;

(ii) $(x \to y) \to y = (y \to x) \to x$;

(iii) $x \to (y \to z) = y \to (x \to z)$.

[6] Tarski's "high-school identity problem": Do these laws imply all laws of $\langle \omega; x+y, xy, x^y, 1\rangle$? This was solved; the answer is negative.

$x+y=y+x$	$xy=yx$	$x+(y+z)=(x+y)+z$	$x(yz)=(xy)z$
$x(y+z)=xy+xz$	$x^{y+z}=x^yx^z$	$(xy)^z=x^zy^z$	$(x^y)^z=x^{(yz)}$
$x\cdot 1=x$	$x^1=x$	$1^x=1$	

[7] Robbins' Problem: Do these laws define Boolean algebras? The answer is "yes"; the proof was found by computer in 1996.

(i) $\vee$ is commutative;

(ii) $\vee$ is associative;

(iii) $((x \vee y)' \vee (x \vee y')')' = x$.

3. Algebras and basic constructions

- A function $f : A^n \to A$ is an *n-ary operation* on A; n is its "arity."

  (For $n = 0, 1, 2, 3$ we say "nullary", "unary", "binary", "ternary".)

- An *algebra* is a set A with a given family of operations f_γ ($\gamma \in \Gamma$), called the "basic operations" of A. Officially, the algebra is $\langle A; f_\gamma, \gamma \in \Gamma\rangle$. Texts often use a separate letter to distinguish the algebra from the set, but we'll follow the informal practice of group theory and use A for both.

- The *type* of $\langle A; f_\gamma, \gamma \in \Gamma\rangle$ is the function $\tau : \Gamma \to \omega$ given by $\tau(\gamma) = n_\gamma$, the arity of f_γ. Two algebras of the same type are *similar*. In discussions involving more than one algebra, we'll normally assume that all the algebras are similar. Usually Γ will be finite; if $|\Gamma| = m$, then it is simplest to choose $\Gamma = 0, \ldots, m-1$ and write the n_γ as a sequence.

  For example, the type of a Boolean algebra $\langle B; \vee, \wedge, 0, 1, ' \rangle$ can be written $\langle 2, 2, 0, 0, 1\rangle$.

- A *subalgebra* of an algebra A is a subset $S \subseteq A$ that is closed under all the basic operations of A.

- An algebra A is said to be *generated* by its elements $g_1, \ldots, g_n$ if the smallest subalgebra of A that contains all the g_i is A itself.

- A *homomorphism* $\phi : A \to B$ between similar algebras is a map compatible with the basic operations of A and B.

- The *direct product* of a family of similar algebras, $A_1 \times A_2$ or more generally $\prod_{\gamma\in\Gamma} A_\gamma$, is the set-theoretic cartesian product with operations computed coordinatewise.

- A *congruence relation* on A is an equivalence relation θ on A that is compatible with the basic operations of A.

- For a congruence relation θ on A, the blocks of θ form an algebra A/θ of the same type, with a natural surjective homomorphism $\eta : A \to A/\theta$.

4. Terms and varieties

- A *term* t or $t(x_1, \ldots, x_n)$ of type τ is a formal expression as a string of symbols, defined recursively as follows, starting from variable symbols $x_1, \ldots, x_n$ for τ:

  (a) Each x_i is a term, and

  (b) if $t_1, \ldots, t_{n_\gamma}$ are terms, so is $\mathbf{f}_\gamma(t_1, \ldots, t_{n_\gamma})$, where $\mathbf{f}_\gamma$ and the commas and parentheses are symbols and $\gamma \in \Gamma$.

  A term t in variable symbols $x_1, \ldots, x_n$ is often described by $t(x_1, \ldots, x_n)$. For algebras with familiar notations we use those notations instead; for example, a group term might be written as $x_1(x_2^{-1}x_3)$.

- For elements $a_1, \ldots, a_n$ of an algebra A and term $t(x_1, \ldots, x_n)$, the *value* $t(a_1, \ldots, a_n)$ is the element of A obtained by using the operations of A while following $t(x_1, \ldots, x_n)$ as a recipe.

  Thus t induces a function on $A^n \to A$. The functions so induced are called the *n-ary term functions* on A. (For polynomial functions, see below.)

- A *term relation* $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$ is an equation holding for a *particular* n-tuple of elements of A.

- A *law* is a formal equation $t_1 = t_2$ or $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$, with $(\forall x_1) \ldots (\forall x_n)$ understood. Many authors write $t_1 \approx t_2$ to distinguish such formal laws from equations involving elements. The law $t_1 = t_2$ *holds* in A when *all* n-tuples $a_1, \ldots, a_n$ from A satisfy the term relation $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$.

  We also say A *satisfies* $t_1 = t_2$, or A is a *model* of $t_1 = t_2$, or write $A \models t_1 = t_2$.

- A *variety* of algebras of a given type is the class of all models of some set of laws. Examples are the varieties of all groups, of all abelian groups, of all lattices, of all distributive lattices, and of other kinds of algebras whose laws are given in §2.

  If A is an algebra we write $\mathrm{Var}(A)$ for the variety determined by all laws holding in A, which is the smallest variety containing A.

- A *polynomial* or *polynomial function* on A is a term function in which some entries may be held constant. For example, if $A = \mathbf{R}$ as a ring, $f(x) = 2x = x + x$ is a term function while $g(x) = \pi x$ is not a term function but is a polynomial function, obtained from the term function $h(x, y) = xy$ by $g(x) = h(\pi, x)$.

5. Some theorems

Familiar theorems from group theory all generalize, except that in groups we focus on normal subgroups, but for algebras in general we focus on congruence relations, a generalization of the coset decomposition of a normal subgroup. The reason is that in groups the whole coset decomposition is determined by knowing the block containing the identity element, while for algebras in general no one block determines the rest.

- The subalgebra of A generated by $g_1, \ldots, g_n$ is the set of elements of the form $t(g_1, \ldots, g_n)$ for some term t in n variables.

- The image of a homomorphism is a subalgebra.

- The set $\mathrm{Con}(A)$ of all congruence relations on A is a lattice, the *congruence lattice* of A.

- For $\theta \in \mathrm{Con}(A)$, the set A/θ of blocks is an algebra, in an obvious way, of the same type as A.

- If $\phi : A \to B$ is a homomorphism, then the equivalence relation on A induced by ϕ is a congruence relation, which we call $\ker \phi$, the *kernel* of ϕ.

  Observe that if $\theta \in \mathrm{Con}(A)$ and $\eta : A \to A/\theta$ is the natural surjection, then $\ker \eta = \theta$.

- If $\phi : A \to B$ is a surjective homomorphism, then $B \cong A/\ker \phi$ (the **first isomorphism theorem**).

  Thus we have an "internal description" of all the homomorphic images of A, up to isomorphism.

- If $\phi : A \to B$ is a surjective homomorphism, then the congruence relations on B correspond one-to-one to the congruence relations on A that contain $\ker \phi$ (the **correspondence theorem**).
- For a direct product $P = \prod_{\gamma \in \Gamma} A_\gamma$, for each $\gamma \in \Gamma$ the coordinate projection $\pi_\gamma : P \to A_\gamma$ is a surjective homomorphism.

6. Problems

Problem U-1. For each of these algebras K, find (i) a 1-variable law of the algebra that does not hold in *all* algebras of the same type, and (ii) (if you can) a law in 2 or more variables that is not an obvious consequence of a 1-variable law of the algebra. No proofs are required.

(a) Perkins' semigroup;

(b) Murskii's 1-binary algebra;

(c) Shallon's graph algebra [note: the operation is idempotent];

(d) the permutation group S_3.

(e) the tournament (8).

(A *tournament* is a directed graph in which every two vertices are joined by a single edge oriented one way or the other. It can be envisioned as a record of who won each match in a "round-robin" tournament, where each player has played every other player once—the arrow points towards the player who won. A tournament can be made into an algebra by letting $x \vee y$ be the winner and $x \wedge y$ the loser of the game between x and y.)

Problem U-2. For the 1-unary algebra $\langle A; f \rangle$ of Example (7), find its equational theory (the set of all laws that hold). You'll need to consider the possibilities $f^n(x) = f^m(y)$ and $f^n(x) = f^m(x)$ $(m \geq n \geq 0)$. Sketch your reasoning.

Problem U-3. For the two-element group $C_2 = \{e, a\}$, invent a procedure for telling whether a given group law holds in C_2. (For example, $((xy)z^{-1})^{-1} = x^{-1}(zy)$?)

Problem U-4. For each of the algebras of examples (4)(for $\mathbf{R}$), (6), (7), (8), (10), (12), (15), (16), (17) in §1, comment on its subalgebras. If there are just a couple, say what they are; if there are many, either describe them all or describe a typical one. No proofs are required.

Problem U-5. Of the binary operations involved in the examples from §1, list those that are *not* commutative.

Problem U-6. Prove the following in an arbitrary lattice L:

(a) If $t(x_1, \ldots, x_n)$ is a lattice term and $I_1, \ldots, I_n$ are ideals of L, then in Ideals(L) we have

$t(I_1, \ldots, I_n) = \{a \in L : a \leq t(i_1, \ldots, i_n)$ for some $i_1 \in I_1, \ldots, i_n \in I_n\}$.

(b) If L obeys a lattice law $t \approx u$, then so does Ideals(L). (Thus Ideals$(L) \in$ Var(L).)