Brief solutions to the Sample Final

NOTE: These are BRIEF versions of solutions. On the exam, you should say more than this in most cases, unless told to be brief.

1. (a) Use ratio test; the series converges (absolutely since the terms are all positive anyway).

(b) Since cosine is bounded, you can see it's really like $\sum_{n=1}^{\infty} \frac{1}{n^2}$, so use comparison by division with this p-series. So, the series in the problem converges (again, absolutely since all terms are positive).

(c) The series converges since it is alternating with terms going to 0 and decreasing in absolute value. To see this, notice that $\log n - \log(n+1) < 0$ and $|\log n - \log(n+1)| = \log(n+1) - \log n = \log \frac{n+1}{n} = \log(1 + \frac{1}{n})$, which decreases and $\to 0$.

The absolute series diverges. One way to see this is to recall that $\log(1+x) = x - \cdots$ so $\log(1+\frac{1}{n})$ is close to $\frac{1}{n}$ as n gets large, which suggests using comparison by division with the harmonic series. You can use l'Hôpital.

But a better way is to realize that the absolute series is

$\sum_{n=1}^{\infty} |\log n - \log(n+1)| = \sum_{n=1}^{\infty} (\log(n+1) - \log n) =$
$(\log 2 - \log 1) + (\log 3 - \log 2) + (\log 4 - \log 3) + \cdots$.

Because of cancellation, the partial sum $s_n = \log(n+1)$, which diverges to infinity.

(d) Converges by the ratio test, again absolutely.

2. Not all of these were discussed in detail in our course. Some that you should be able to do with what you know are these:

(1) True; this is just a statement of the Weierstrass M-test.

(3) False; remember, if a series converges then its terms go to 0 but not vice-versa; a counterexample is the harmonic series.

(4) False; this just says the terms are decreasing in absolute value, so again the harmonic series would be a counterexample.

(5) True; there were some homework problems based on this principle.

Some others that are doable from what you know but that might be hard to see under pressure are these:

(2) False; a counterexample would be to have all the f_n be constant functions with values equal to the alternating harmonic series. The fact that

the functions are constant makes uniformity automatic, and the alternating harmonic series converges but not absolutely.

(6) False; it sounds logical but in lecture once I gave an example of a function that is so shallow near $x = 0$ that all its derivatives are 0 there and yet the function is nonzero except at $x = 0$, so its Maclaurin series doesn't converge to it except at $x = 0$. The example was $f(0) = 0$, $f(x) = e^{-1/x^2}$ otherwise.

(9) True; this is a restatement of what uniform convergence means.

(10) False; you could just take $v_n = -1$ for all n. But maybe they meant $|u_n| > |v_n|$; in that case it's still false, since the u_n could be the alternating harmonic series and the v_n could be half the positive harmonic series.

3. Notice this question is about uniformly convergent *series* of functions, so try the Weierstrass M-test. The idea is simply to see if you can bound the functions with a convergent series of *numbers* not depending on x.

(a) Yes; take $M_n = \frac{1}{e^n}$. This is convergent geometric, with ratio $r = \frac{1}{e}$.

(b) Yes; notice that the value of each term is largest for $x = 0$, so take $M_n = \frac{1}{n^2}$.

4. This is somewhat like p. 145, Ex. 16. You need to reason backwards a bit until you get to a series you can deal with.

The ratio test will show it converges for all x.

Only even powers of x are involved so put $g(x) = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$, giving $f(x) = g(x^2)$. Notice that $g(x)$ is the derivative of $\sum_{n=0}^{\infty} \frac{(x^{n+1})'}{n!} = (\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!})' = (x \sum_{n=0}^{\infty} \frac{(x^n)}{n!})' = (xe^x)' = (1+x)e^x$.

Then $f(x) = (1+x^2)e^{x^2}$. [Solution due to Vrej.]

5. f is an even function so $b_n = 0$ for all n.

$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \cdots = \frac{2}{3}\pi^2$.

$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx = \cdots \text{ (by parts) } = \frac{4(-1)^n}{n^2}$.

Because f is continuous and piecewise smooth, the Fourier series of f converges to f at all x. At $x = \pi$ we get

$\pi^2 = \frac{1}{2}\frac{2}{3}\pi^2 + \sum_{n=0}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi$. Since $\cos n\pi = (-1)^n$ (something you should know), we get $\pi^2 = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$, which gives $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

6. (a) Notice that this is a power series centered about $x = 2$, so you expect that the answer will be an interval symmetric about $x = 2$ (except possibly for the end points). Use the ratio test. You should get $R = \frac{1}{3}$. At $x = 2 \pm \frac{1}{3}$, you get series that in absolute value can be compared by division to $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is convergent, so the answer is $[\frac{5}{3}, \frac{7}{3}]$.

(b) The ratio test will give $R = 1$. At the endpoints $x = \pm 1$, the series diverges since its terms are unbounded. So the answer is the open interval $(-1, 1)$.

7. (a) Expand $(1 + x)^{\frac{3}{2}}$ with a binomial series and put x^2 for x to get the Maclaurin series of $(1 + x^2)^{\frac{3}{2}}$. You also know the series for $\cos x$. Multiply the two series. Since only even powers of x are involved, your answer will stop with the x^4 term.

(b) Take 1 plus the series for e^x minus the series for $\sin x$ times itself.

8. Notice that this involves uniform convergence of a *sequence* of functions.

All f_n $(n = 1, 2, \ldots)$ have value 0 at the endpoints. For any other fixed value of x, $nx^n \to 0$ as $n \to \infty$ (by l'Hôpitcal). Therefore $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all x.

Find $\|f_n - f\|$ (or equivalently, the max on the interval) by checking the value of f at endpoints and critical points; you get $x = \dfrac{n}{n+2}$ (which is in the interval) and

$\|f_n - f\| = n(\frac{n}{n+2})^n(1 - \frac{n}{n+2})^2$,

$\leq 1^n \, n(\frac{2}{n+2})^2 \to 0$. Therefore the series is uniformly convergent. [Solution adapted from Vrej.]