Solutions to Midterm Exam #1, Version A

If you have a question about the grading of a problem, please write it on the front of the exam and hand the exam back in during the next several days. Do not write on the inside of the exam.

The median score was 31. The midterm score is recorded but not a midterm letter grade. Here are rough equivalents, though: 20-29: C– to C+; 30-39: B– to B+; 40-50: A– to A+.

Solutions:

1. (a) diverges, comparison with the harmonic series, or comparison by division with the harmonic series.

(b) converges; ratio test is easiest.

(c) converges; ratio test or comparison with geometric series of ratio s where $\frac{2}{3} < s < 1$. (You can choose a specific s.)

(d) converges; integral test. Notice the misprint; the sum should start at $n = 2$.

(e) converges; comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.05}}$.

2. (a) converges absolutely since abs terms give p-series, $p > 1$. The alternating series theorem also applies.

(b) converges conditionally: the alternating series theorem applies, so it converges; abs values give a divergent p-series with $p = \frac{1}{2} < 1$.

(c) converges absolutely by comparison of abs values with p-series, $p = 2$, since $|\cos n| \le 1$.

(d) terms don't go to 0, so diverges.

(e) converges absolutely by the ratio test applied to the absolute terms.

3. (a) $f(x) = \dfrac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} 3^n x^n$ since geometric, $r = 3x$. Converges for $|r| < 1$, so for $|x| < \frac{1}{3}$, so radius of convergence is $\frac{1}{3}$.

(b) To do from Taylor/Maclaurin as requested: Derivs of f are e^{-3x}, $-3e^{-3x}$, $9e^{-3x}$, etc., so series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!} x^n$. This converges for all x since ratio of consecutive terms is $-\dfrac{3}{n+1} \to 0$.

(c) $f'(x) = \dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$ (geometric). Integrating term by term, we get $f(x) = C + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ (re-indexing). Setting $x = 0$ we see $C = 0$. The series converges for $|x| < 1$ by the ratio test, so the radius of convergence is 1. (Can also use Taylor/Maclaurin directly. No points off for forgetting C, this time.)

4. Here $f(x) = x^{\frac{5}{4}}$, $a = 1$, $b = 1.1$. We have $f'(x) = \frac{5}{4}x^{\frac{1}{4}}$, $f''(x) = \frac{1}{4}\frac{5}{4}x^{-\frac{3}{4}}$, $f'''(x) = -\frac{3}{4}\frac{1}{4}\frac{5}{4}x^{-\frac{7}{4}}$. Taylor says $f(x) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(a)(b-a)^2 + R_2$, where $R_2 = \frac{1}{6}f'''(\xi)(b-a)^3$ for some ξ between a and b. In the present example this says

$f(x) = 1 + \frac{5}{4}(.1) + \frac{1}{2}\frac{1}{4}\frac{5}{4}(.1)^2 + R_2$, where

$|R_2| = \frac{1}{6}\frac{3}{4}\frac{1}{4}\frac{5}{4}\xi^{-\frac{7}{4}}(.1)^3 \leq \frac{1}{6}\frac{3}{4}\frac{1}{4}\frac{5}{4} \cdot 1 \cdot (.001)$ (since $\xi^{-\frac{7}{4}}$ is a decreasing function of ξ in $[1, 1.1]$). This last expression is less than .001, as required.

5. (a) $\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \dfrac{1}{n} - \log N$. Notice that the limit applies to both the sum and the log.

This expresses the fact that the N-th partial sum of the harmonic series is larger than $\log N$ by an amount with a finite limit, even though the harmonic series is divergent and $\log N$ also goes to infinity. It is not correct to write $\sum_{n=1}^{\infty} \dfrac{1}{n}$ instead, because that sum doesn't have a finite value. It is also not correct to mix up the roles of n and N.

Some people didn't realize this question might be on the exam, but it was part of homework problems and in lecture 4-F it was specifically mentioned as something to know.

(b) (i) Let $s_n = a_1 + \cdots + a_n$. Then $s_n \to s = \sum_{n=1}^{\infty} a_n$, by definition. Also $s_{n+1} \to s$. Then $a_{n+1} = s_{n+1} - s_n \to s - s = 0$ since limits of sequences are compatible with addition. Therefore $a_n \to 0$.

(ii) Since $\frac{a_n}{b_n} \to C > 0$, eventually $\frac{a_n}{b_n} \leq 2C$, say for all $n \geq N$. Then for this tail, $a_n \leq 2Cb_n$, also a convergent series, and so $\sum_{n=1}^{\infty} a_n$ converges by comparison.

(iii) Choose any s with $r < s < 1$. Since $\frac{a_{n+1}}{a_n} \to r < s$, eventually $\frac{a_{n+1}}{a_n} \leq s$, say from the N-th term on. Then we have $a_{N+1} \leq sa_N$, $a_{N+2} \leq sa_{N+1} \leq s^2 a_N$, and so on, giving $a_{N+k} \leq s^k a_N$ for each $k = 0, 1, 2, \ldots$. Then the series $a_N + a_{N+1} + \cdots$ converges by comparison with the convergent geometric series $a_N + sa_N + s^2 a_N + \cdots$. Finally, since a tail of the series convergenes the whole series $\sum_{n=1}^{\infty} a_n$ converges.