Notes on series

1. General

Definition. A series $a_1+a_2+\cdots$ *converges* with sum S, written symbolically as $a_1+a_2+\cdots=S$ or $\sum_{n=1}^{\infty} a_n = S$, when its sequence of partial sums converges to S.

Definition. If you delete the first N terms of a sequence $a_1, a_2, \ldots$ (for some N), the sequence of remaining terms $a_{N+1}, a_{N+2}, \ldots$ is called a *tail* of the original sequence.

Convergence of the whole sequence is equivalent to convergence of any tail. For example, $a_1, a_2, \cdots$ converges $\Leftrightarrow$ $a_{100}, a_{101}, \ldots$ converges[1]; and in that case, both converge to the same value.

Similarly, you can take a tail of a series, but for a convergent series the sum of a tail will be different because terms have been omitted.

2. Rates of growth

Some functions in order of increasing rates of growth are these:

$$\log\log n,\quad \log n,\quad n^{.01},\quad \sqrt{n}=n^{.5},\quad n,\quad n^2,\quad n^3,\quad 1.01^n,\quad 2^n,\quad e^n,\quad 3^n,\quad n!,\quad n^n$$

For now you can take it for granted that each divided by the next goes to 0, and of course the same is true for similar functions not listed. For example, $\frac{n^3}{1.01^n} \to 0$ as $n\to\infty$, even though n^3 is larger than 1.01^n at first:

n :	1,	2,	3,	$\cdots$	300,	$\cdots$	2339,	$\cdots$	5000,	$\cdots$
$\frac{n^3}{1.01^n}$:	0.99,	7.84,	26.20,	$\cdots$	1364431.16,	$\cdots$	0.99,	$\cdots$	0.0000003,	$\cdots$

To see why, take logs: $\log\frac{n^3}{1.01^n} = 3\log n - n\log 1.01$; n eventually overwhelms $\log n$ even though $\log 1.01 \approx 0.01$. Later we'll use §3.1 of the text.

3. Series of positive terms

Theorem. (Boundedness test) A series of positive terms converges $\Leftrightarrow$ its partial sums are bounded.

[1] $\Leftrightarrow$ is read "if and only if" and means that if either side is true, so is the other

Brief reason: The partial sums form an increasing sequence bounded above, and such a sequence must converge.

3.1 Example: $1.01001000100001\cdots$ (where there are 1, 2, 3, ... zeros between ones) means $1 + .01 + .00001 + .0000000001 + \cdots$, which converges to some number since all partial sums are < 2. (Of course, this is no surprise since we're used to the idea that every infinite decimal expression gives a number.)

Theorem. (Comparison test) If $0 \leq a_n \leq b_n$ for each $n = 1, 2, \cdots$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Brief reason: The partial sums $a_1 + \cdots + a_n$ are bounded by $\sum_{n=1}^{\infty} b_n$.

3.2 Examples: (a) $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$; (b) $\sum_{n=1}^{\infty} \frac{2n}{2n^2 - 1}$.

In this second example we use the test in reverse to show that if the series of smaller terms diverges then so does the series of larger terms[2].

Theorem. (Integral test) If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and $a_n = f(n)$ for some decreasing continuous function $f(x)$, then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \int_1^{\infty} f(x)\, dx$ converges.

Intuitive reason: From pictures, the integral is between $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$.

3.3 Examples: (a) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. (For which values of p does it converge?) (b) $\sum_{n=1}^{\infty} \frac{1}{n \log n}$, $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$.

Theorem. (Comparison by dividing [not in the text]) (1) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms and $\frac{a_n}{b_n} \to C$ as $n \to \infty$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

To keep this straiht, think: $\frac{\text{unknown}}{\text{known}} \to$ some number.

Brief reason: In the case $C \neq 0$, eventually $\frac{a_n}{b_n} < 2C$ so $a_n < 2Cb_n$. In the case $C = 0$, eventually $\frac{a_n}{b_n} < 1$ so $a_n < b_n$.

Careful: C is a number, not ∞. Also, notice that if $C \neq 0$ then the test works the other way around also, so $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=1}^{\infty} b_n$ converges.

3.4 Examples: (a) $\sum_{n=1}^{\infty} \frac{n+2}{n^3+4}$; (b) $\sum_{n=1}^{\infty} \frac{n+2}{n^2+3}$; (c) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ again.

[2] This is the "contrapositive" form: "if P then Q" has contrapositive "if not Q then not P", which is really the same thing.

Theorem. (Ratio test) Suppose $\sum_{n=1}^{\infty} a_n$ $(n = 1, 2, \cdots)$ is a series of positive terms and $\frac{a_{n+1}}{a_n} \to r$ as $n \to \infty$.

$$\begin{cases} \text{if } r < 1 & \text{the series converges} \\ \text{if } r > 1 & \text{the series diverges} \\ \text{if } r = 1 & \text{can't tell} \end{cases}$$

Intuitive reason in the case $r < 1$: As the ratio of consecutive terms gets close to r, the series looks pretty much like a convergent geometric series with ratio r.

Better reasoning: In some tail, we can do an actual term-by-term comparison to a geometric series with ratio slightly higher than r but still less than 1.

3.5 Examples: (a) $\sum_{n=1}^{\infty} \dfrac{1}{2^n + 1}$ again; (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$; (c) $\sum_{n=1}^{\infty} \frac{1}{n!}$; (d) $\sum_{n=1}^{\infty} \dfrac{n^3}{1.01^n}$.

3.6 Examples with $r = 1$ (can't tell): (e) $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series), (f) $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$.

Notes:

(1) Don't confuse the ratio test with comparison by dividing; both involve ratios but they're ratios of different things.

(2) In the text, the ratio test appears in the section on series with positive and negative terms, but using absolute values, so the idea is the same as here.

4. Series of positive and negative terms

In other words, this section is about *any* series of nonzero terms.

Definition. $\sum_{n=1}^{\infty} a_n$ **is absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

4.1 Examples: (a) The geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$; (b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$.

4.2 Non-example: The alternating harmonic series (c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

Theorem. An absolutely convergent series is convergent.

Brief reason: Look at the series consisting just of positive terms and at the series consisting just of negative terms. They both converge.

4.3 Example: If you take a convergent geometric sequence and change the signs of some terms at random, you get a convergent sequence.

Definition. An **alternating series** is a series where the terms are alternately positive and negative.

Theorem. An alternating series must converge if its terms are decreasing in absolute value, with limit 0.

4.4 Example: The alternating harmonic series, which converges to $\log 2$.

More: For an alternating series fitting the theorem, with sum S, the partial sums are alternately larger than S and smaller than S.

Note: A series that is convergent but not absolutely convergent is called *conditionally convergent*. Think of a conditionally convergent series as converging delicately, while an absolutely convergent series converges more robustly. (See below.)

5. Sidelights[3]

(1) If you take an absolutely convergent sequence and rearrange the order of the terms the result is still absolutely convergent with the same sum as before. But if you take a conditionally convergent sequence, by rearranging the order of the terms you can make a new sequence that converges to any number you want!

(2) **Stirling's Formula** says that $n! \approx \frac{n^n}{e^n}\sqrt{2\pi n}$, in the sense that the ratio of this expression to $n!$ converges to 1. For $n = 10$ the expression is already within 1% of 10!, and it gets better and better percentagewise for larger n.

(3) The integral test shows that $\sum_{n=1}^{N} \frac{1}{n}$ is close to $\log N$. Even though both go to infinity as n gets large, their difference has a limit γ (gamma) $= 0.5772156649\cdots$, called **Euler's constant** (pronounced "oiler"):

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} \right) - \log N.$$

No one knows whether γ is a rational number (fraction of integers) or not.

(4) The Riemann (ree-mahn) zeta function is defined by $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. It is involved in the Riemann Hypothesis, one of the most famous unsolved mathematical conjectures.

[3] You are not responsible for these unless told to be later.