Some problems

(To be done when they are assigned.)

1. Problems relating to Handout B

Problem C-1. In (7), what are the first five terms you get (in decimals) if you start with $a_1 = 2$ and use the same iteration?

(Compare your results with the values in Section 4 of Handout B.)

Problem C-2. In (7), assuming there *is* a limit, show that this limit must be $\sqrt{5}$. (Method: Suppose that $a_n \to L$. Then let $n \to \infty$ in the equation $a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right)$ by putting L wherever you see a_n or a_{n+1}, and solve for L. Could the limit be negative?)

Problem C-3. This problem shows why the sequence converges to $\sqrt{5}$ so fast. Let $E_n = a_n - \sqrt{5}$, the "error" of the n-th term. Show that $E_{n+1} = \dfrac{E_n^2}{2(E_n + \sqrt{5})}$. (So if E_n is already small, then E_{n+1} is much smaller.)

Method: Calculate E_{n+1} in terms of E_n by starting from $a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right)$, substituting $a_n = E_n + \sqrt{5}$ and $a_{n+1} = E_{n+1} + \sqrt{5}$ and solving for E_{n+1}.

Problem C-4. In Section 3 of Handout B, these equations mean that the function on the left equals the sum of the series on the right for each number x in the range given.

A correction: (24) is valid for $|x| \leq 1$, not just $|x| < 1$. This has been fixed in the on-line version.

(a) Explain (13) using one of (17)–(24). (Choose a particular number for x, making sure it's a value for which the series converges so the equation applies.)

(b) Explain (14) using one of (17)–(24).

(c) Explain (15) using one of (17)–(24).

Problem C-5. (a) Explain how you know (17) is true, for the range of x given. (Geometric series!)

(b) Explain how you know (18) is true. (Still geometric. What is the ratio?)

(c) Explain how you know (19) is true. (Still geometric. What is the ratio? Is it between -1 and 1?)

Problem C-6. For approximations, it is helpful to use the first few terms of a power series. (This is the same as using a Taylor polynomial.) A "third order approximation" means to use terms through the x^3 term. (If there is no x^3 term, the coefficient of x^3 is 0, so there's less work to do.)

In each of the following cases, use a third-order approximation and compare it with the value you get using a hand calculator. For trig functions, set it to radians, and for logs use natural logs!

(a) Use (20) to approximate $e^{0.2}$.

(b) Use (21) to approximate $\cos(0.1)$.

(c) Use (22) to approximate $\sin(0.1)$.

(d) Use (24) to approximate $\arctan(0.1)$.

(e) Use (23) to approximate $\log(1.2)$.

Problem C-7. See if you can get one of the sums of series (8) through (16), after using some algebraic manipulations and knowledge of trigonometry,

(a) by setting $x = \frac{1}{2}\pi$ in (25);

(b) by setting $x = \frac{1}{2}\pi$ in (26);

(c) by setting $x = \pi$ in (26).

Problem C-8. When we get to power series you'll see that they work just as you'd expect with regard to differentiation and integration; you can differentiate or integrate "term by term".

Differentiate both sides of each of the following equations and see if you get one of the same group of equations (say which).

(a) (22); (b) (23); (c) (24); (d) (20).

2. Problems on limits of sequences

Problem C-9. For the sequence $a_1, a_2, a_3, \cdots$ with $a_n = \dfrac{n}{2n+1}$, say what its limit L is and find a value of N so that the distance $|a_n - L| < .01$ for all $n \geq N$.

(Method: Once you know what number L is, write down $L - .01 < a_n < L + .01$ and mess around. In this problem only one side will matter.)

Problem C-10. For the sequence $a_1, a_2, a_3, \cdots$ with $a_n = \dfrac{n-1}{n+1}$, say what its limit L is and find a value of N so that the distance $|a_n - L| < .001$ for all $n \geq N$.