Categories and Stone duality

1. Motivation

In the first part of the 20th century people concentrated on the study of individual spaces, such as groups, topological spaces, and various kinds of linear spaces. Interesting associations between spaces were found: homology and homotopy groups associated with topological spaces, topological spaces associated with Boolean algebras, etc. However, it was eventually realized that maps between spaces are equally important, and that many properties of maps are best viewed in a more general framework.

Some examples that such a framework should cover:

(1) the class of groups and homomorphisms between them;

(2) the class of topological spaces and continuous functions between them;

(3) the class of partially ordered sets and isotone maps between them.

2. Categories

Definition. A *category* consists of

(i) a class $\mathcal{C}$ (whose members are called *objects*), together with

(ii) for each $A, B \in \mathcal{C}$, a set $\mathrm{Morph}(A, B)$ (whose members are called *morphisms*), and

(iii) for each $A, B, C \in \mathcal{C}$, an operation $\circ$ between morphisms, so that for $\phi \in \mathrm{Morph}(A, B)$ and $\psi \in \mathrm{Morph}(B, C)$, $\phi \circ \psi \in \mathrm{Morph}(A, C)$ is defined,

with the properties

(a) $\phi \circ (\psi \circ \eta) = (\phi \circ \psi) \circ \eta$ for morphisms between successive objects, and

(b) for each $A \in \mathcal{C}$ there is an element $\mathbf{1}_A \in \mathrm{Morph}(A, A)$ such that $\mathbf{1}_A \circ \psi = \psi$ and $\phi \circ \mathbf{1}_A = \phi$ for all $\psi \in \mathrm{Morph}(A, B)$ and $\phi \in \mathrm{Morph}(B, A)$, for any $B \in \mathcal{C}$.

In most familiar examples, the objects are sets, the morphisms are functions, and $\circ$ is composition (written composing to the right). The examples (1), (2), (3) are of this kind, but not all useful categories are.

We usually say "the category $\mathcal{C}$" with the associated morphisms and composition understood.

Exercise. Define what it should mean for two objects in a category to be isomorphic. *Solution.* $A \cong B$ means there exist morphisms $\phi \in \mathrm{Morph}(A, B)$ and $\psi \in \mathrm{Morph}(B, A)$ such that $\phi \circ \psi = \mathbf{1}_A$ and $\psi \circ \phi = \mathbf{1}_B$.

3. Functors

There are many examples in mathematics where from one object A one can construct another kind of object $F(A)$. For example, with a Boolean algebra there is associated a topological space (its prime ideal space), and with a topological space there is associated its homology group. These constructions extend to maps between the objects and have reasonable properties for maps. However, upon closer examination it can be seen that there are really two kinds:

Definition. For categories $\mathcal{C}$ and $\mathcal{D}$, a *covariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of

(i) for each object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$, and

(ii) for each $A, B \in \mathcal{C}$ and morphism $\phi \in \mathrm{Morph}(A, B)$, a morphism $F(\phi) \in \mathrm{Morph}(F(A), F(B))$,

with the properties

(a) $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ (whenever $\circ$ makes sense) and

(b) $F(\mathbf{1}_A) = \mathbf{1}_{F(A)}$ for each $A \in \mathcal{C}$.

An example is the fundamental-group functor π_1. Here $\mathcal{D}$ is the category of groups; the objects of $\mathcal{C}$ are pairs (X, p) consisting of a topological space X and a designated point in X, and the morphisms of $\mathcal{C}$ are continuous functions that take designated points to designated points. $F(X, p)$ is the fundamental group $\pi_1(X, p)$ consisting of equivalence classes of loops at p in X.

A *contravariant functor* is defined similarly, except that in (ii) we have $F(\phi) \in \mathrm{Morph}(F(B), F(A))$ and in (a) $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$. An example is the prime-ideal-space functor from Boolean algebras to topological spaces.

For a functor F from $\mathcal{C}$ to $\mathcal{D}$ and a functor G from $\mathcal{D}$ to $\mathcal{E}$, the result of following F by G is again a functor, which will be covariant if both F and G are covariant or both F and G are contravariant.

4. Natural duality

One advantage of the category-theoretic point of view is that it becomes possible to define concretely what it means for isomorphisms to be "natural". The idea is to look at isomorphisms of whole categories at a time, instead of individual objects; the naturalness consists of compatibility with morphisms between objects.

There are two concepts: natural isomorphism of two categories, defined with covariant functors, and natural duality, defined with contravariant functors. Anticipating Boolean duality, let's concentrate on the second kind.

Definition. A *natural duality* between categories $\mathcal{C}$ and $\mathcal{D}$ consists of

(i) a contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ and a contravariant functor G from $\mathcal{D}$ to $\mathcal{C}$,

(ii) for each $A \in \mathcal{C}$, an isomorphism α_A of A with $G(F(A))$ and for each $B \in \mathcal{D}$, an isomorphism β_B of B with $F(G(B))$, such that

(a) for each $A_1, A_2 \in \mathcal{C}$ and morphism $\phi \in \mathrm{Morph}(A_1, A_2)$, the following diagram is commutative, i.e., the same morphism is obtained by taking the composition along either route from the upper left corner to the lower right:

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_{A_1}} & G(F(A_1)) \\ \phi \downarrow & & \downarrow G(F(\phi)) \\ A_2 & \xrightarrow{\alpha_{A_2}} & G(F(A_2)) \end{array}$$

and (b) for each $B_1, B_2 \in \mathcal{D}$, a similar diagram is commutative.

5. Boolean Duality

This is the case where $\mathcal{C}$ is the category of Boolean Algebras with Boolean homomorphisms and $\mathcal{D}$ is the category of Boolean spaces with continuous functions. For a Boolean algebra A, $F(A)$ is $\Pi(A)$. For a Boolean space X, $G(X)$ is $\mathrm{Clopen}(X)$. For a homomorphism $\phi : A_1 \to A_2$ of Boolean algebras, $F(\phi)$ is the function $\phi^\star : \Pi(A_2) \to \Pi(A_1)$ given by $\phi^\star(\mathcal{P}) = \phi^{-1}(\mathcal{P})$. For a continuous function $f : X \to Y$ between Boolean spaces, $G(f)$ is the map $f^\dagger : \mathrm{Clopen}(Y) \to \mathrm{Clopen}(X)$ given by $f^\dagger(C) = f^{-1}(C)$.