Modular lattices

1. Definition

Recall that a *modular lattice* is a lattice that obeys the modular law

(M) $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$, or equivalently,

(M′) $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge (z \vee x))$.

2. Examples

1. any distributive lattice;

2. Subgp(A) for any abelian group A;

3. Normal(G) for any group G (Dedekind);

4. Subsp(V) for any vector space V;

5. the lattice of flats of any projective plane or projective geometry;

6. any sublattice, product, homomorphic image, or dual of a modular lattice;

7. Ideals(L) for any modular lattice L.

3. Modularity for arbitrary lattices

Theorem. The following conditions on a lattice L are equivalent.

(1) L obeys the modular condition $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$.

(2) L obeys the law $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge (x \vee z))$.

(3) L has no sublattice isomorphic to N_5.

(4) Transposed intervals of L are isomorphic under the obvious maps up and down.

(5) Any three elements of L generate a distributive sublattice provided two of them are comparable.

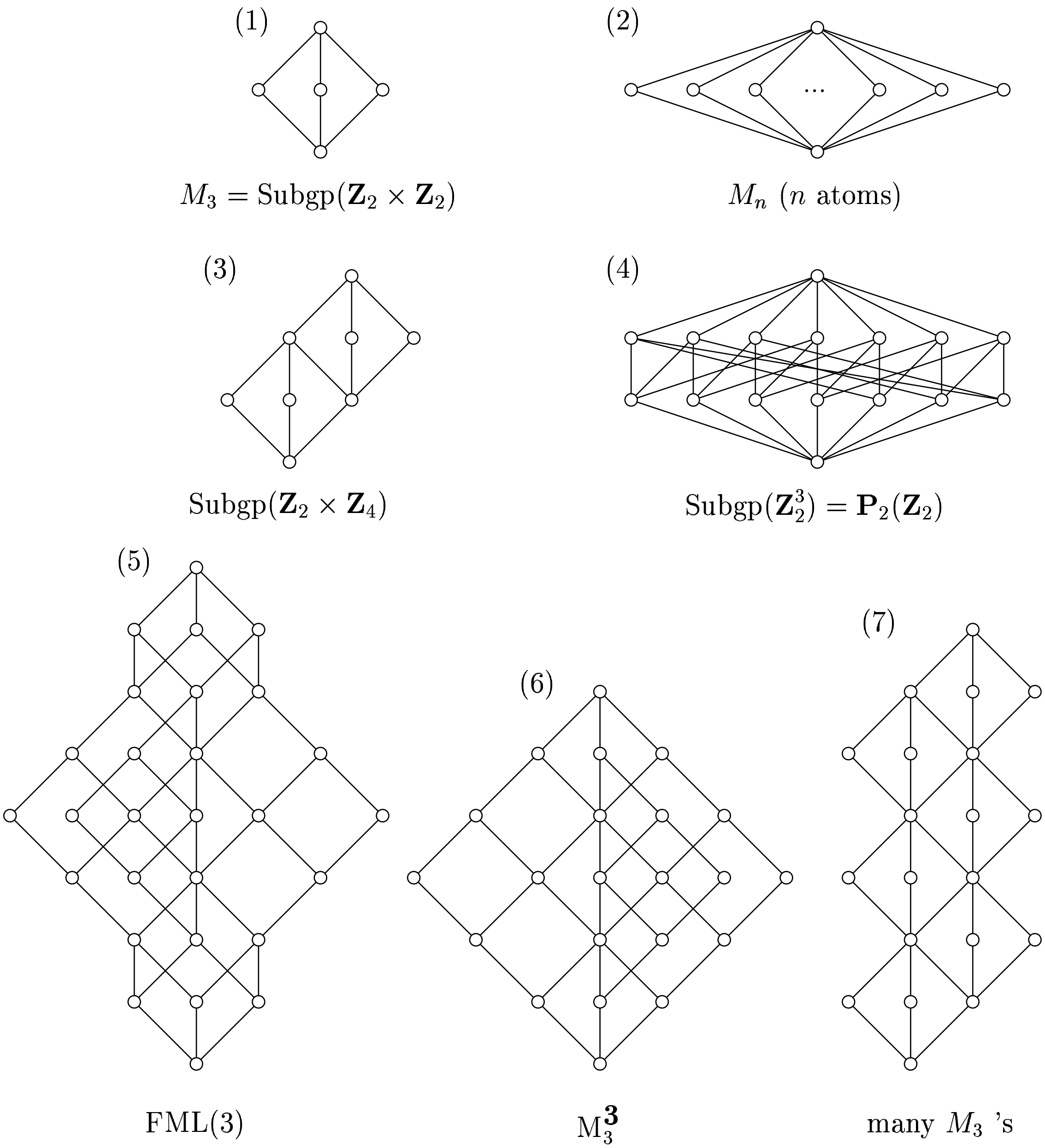

Figure 1: Some modular lattices

4. Modularity for lattices of finite length

Theorem. The following conditions on a lattice L of finite length are equivalent to conditions (1)-(5) above.

(6) In L, a and b cover $a \wedge b \Leftrightarrow a \vee b$ covers a and b.

(7) The height function h in L obeys $h(a \vee b) + h(a \wedge b) = h(a) + h(b)$.

Theorem. Any modular lattice of finite length obeys the Jordan-Dedekind chain condition:

In any interval $[a, b]$, any two maximal chains have the same length.

Remarks.

(i) This last theorem is actually true for any "semimodular" lattice of finite length. A lattice of finite length is *semimodular* if it obeys "$\Rightarrow$" of condition (4).

(ii) The condition (J-D) shows that the height function h is a well-behaved rank function, in that for $a \leq b$, b covers a if and only if $h(b) = h(a) + 1$.

5. Other facts

Proposition **A**. Normal(G) is modular for any group G.

Proposition **B**. In a modular lattice, the sublattice generated by two chains is distributive.

6. Problems

Problem O-1. In Example (5) of Figure 1, how many copies of M_3 can you find?

Problem O-2. In Example (7) of Figure 1, see how long a string of intervals you can find such that each two consecutive ones are transposes and there are no repeats.

Problem O-3. Prove that a lattice L is modular if and only if transposed intervals are isomorphic under the obvious maps up and down: $[a \wedge b, a] \cong [b, a \vee b]$ under $x \mapsto x \vee b$ with inverse $x \mapsto x \wedge a$.

Problem O-4. Let G be a finite abelian group. Answer the following with brief proofs.

(a) Show that if Subgp(G) has a single co-atom, then G is cyclic. (Examine a group element not in the co-atom. Is the converse true?)

(b) Show that if Subgp$(G) \cong \mathbf{n}$ then $G \cong \mathbf{Z}_{p^{n-1}}$ for some prime p.

(c) Show that if Subgp$(G) \cong M_n$, the lattice of length 2 with n atoms, for $n > 1$, then $G \cong \mathbf{Z}_p \times \mathbf{Z}_p$ and $n = p + 1$, or else $G \cong \mathbf{Z}_p \times \mathbf{Z}_q$ and $n = 2$, where p, q are prime, $p \neq q$. (Recall that direct-product decompositions of G into two factors correspond to pairs of complementary subgroups, a lattice-theoretic idea.)

(d) For each subgroup diagram in Figure 2, identify the corresponding finite abelian group (if any). You need explain only why no finite abelian group other than yours fits, not why yours does have the diagram given.

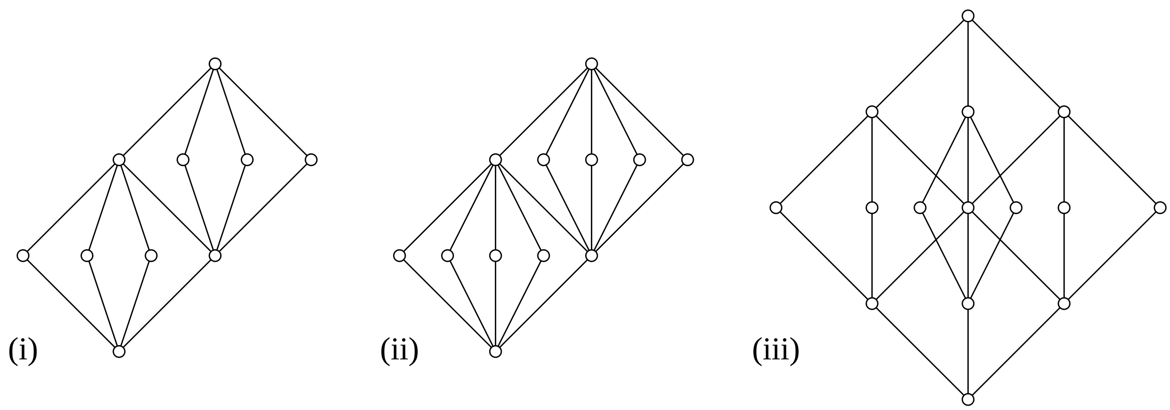

Figure 2: subgroup lattices