An overview of topological spaces

1. Facts to recall from metric spaces

First, recall that in a metric space, the family of open sets is closed under finite intersections and arbitrary unions.

Second, recall that in a metric space, a number of concepts can be characterized in terms of open sets (where as usual, we often refer to subsets as simply “sets”):

- A set is closed ⇔ its complement is open.
- A function is continuous ⇔ the inverse image of every open set is open.
- A space is connected ⇔ it cannot be partitioned into two open sets.
- A space is compact ⇔ every open cover (family of open sets whose union is the space) has a finite subcover (finite subfamily whose union is the space).
- A point is isolated ⇔ it is open as a singleton.
- A sequence converges to x ⇔ for each open set O containing x, the sequence is eventually in O.

2. Topological spaces

These facts suggest that open sets could be used as a starting concept. The resulting theory is more general than the theory of metric spaces and is helpful for various spaces of functions that are not metric.

Definition. A *topology* on a set X is a family $\mathcal{T}$ of subsets of X such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The members of $\mathcal{T}$ are called *open sets* (or simply *open sets*) and X with T is a *topological space.*

We would also list that the the empty set and X itself are open, but these facts already follow from the definition since the union of no sets is empty and the intersection of no subsets is X.

All the concepts from Section 1 can be turned into definitions. Many theorems that you know from metric spaces remain valid. However, the definition of a topological space is mismatched with metric spaces in two ways:

- The definition of a topological space is weak, in that it does not ensure that open sets can distinguish between points. In fact, given any set X we can define a rudimentary topology in which the only open sets are the empty set and X; in this topology, it is no longer true that a convergent sequence has a unique limit, for example. Therefore we often go beyond the definition of a topological space and impose separation conditions, as in §3 below.

- Sequences (indexed by nonnegative integers), which are sufficient to determine the topology in a metric space, are inadequate to do so for topological spaces in general. In other words, it is not true in general that a subset S is closed when any convergent sequence in S converges to a point of S. Instead, one can use "nets"—generalized sequences indexed by a directed set instead. However, theorems in topology can usually be proved without using nets.

3. Separation conditions

Here are some separation conditions of increasing strength. Which (if any) is needed depends on the application.

T_0 space: The open sets distinguish points. In other words, given two points in X, there is an open set that contains one point but not the other.

T_1 space: The open sets distinguish points both ways around: Given two points, for each point there is an open set that contains it but not the other point. An equivalent condition is that every singleton is a closed set.

T_2 space, or Hausdorff space: Any two points can be separated by disjoint open sets. In other words, given two points x, y, there are disjoint open sets O_x, O_y with $x \in O_x$ and $y \in O_y$.

There are stronger possibilities as well.

4. Remarks

- Any metric space is a Hausdorff space.

- A frequent object of study is compact Hausdorff spaces, which include the compact metric spaces such as the real unit interval.

- Just as for metric spaces, for a subset S of a topological space X, we can relativize to S: The *relative topology* on S consists of the intersections of S with the open sets of X.

- Two topological spaces are said to be *homeomorphic* if they are in one-to-one correspondence by a correspondence that preserves open sets. Equivalently, there is a pair of inverse continuous functions between them.

5. Problems

Problem M-1. Show that the two descriptions of a T_1 space are equivalent.

Problem M-2. Consider the metric space $X = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} \cup \{1\} \subseteq \mathbb{R}$. Describe the open sets of X. Which sets are clopen (simultaneously closed and open)?

Problem M-3. Given a topological space X with topology $\mathcal{T}$, a family $\mathcal{U}$ of sets of X is a *base* for the topology if every member of $\mathcal{T}$ is a union of members of $\mathcal{U}$ (possibly infinitely many). If X is the real unit interval with the usual topology, are the open intervals with rational endpoints a base for the topology of X?

Problem M-4. A *Boolean* topological space is a compact Hausdorff space in which the clopen sets form a base for the topology. (For definitions, see problems 2 and 3.) Show that the only connected subsets of a Boolean topological space are singletons. (A subset is said to be connected if it is connected in the relative topology.)

Problem M-5. Show that if a compact Hausdorff space X has the property that every connected subset is a singleton, then X is Boolean.

Problem M-6. Let ω_1 be the "first uncountable ordinal", an uncountable well ordered set for which all principal ideals are countable. Let ω_1^+ be ω_1 with a top element added. Give ω_1^+ the topology for which the open intervals ($\{x : a < x < b\}$ or $\{x : a < x\}$ or $\{x : x < b\}$) form a basis.

(a) Is ω_1^+ compact Hausdorff with this topology?

(b) Are sequences adequate to determine whether a subset is closed?