Lattice ideals

1. Ring ideals

Let R be a commutative ring with 1. As you know,

(1) an *ideal* of R is a nonempty subset A of R such that $a_1 + a_2 \in A$ for all $a_1, a_2 \in A$, and $ra \in A$ for all $a \in A, r \in R$;

(2) an ideal A is *principal* if $A = (a) = \{ra | r \in R\}$ for some $a \in A$;

(3) an ideal A is *proper* if $A < R$.

(4) a ideal A is *prime* if A is proper and $xy \in A$ implies $x \in A$ or $y \in A$;

(5) an ideal A is *maximal* (meaning maximal proper) if A is proper and there is no ideal I with $A < I < R$.

2. Lattice ideals

The concepts are the same, with $\vee$ for $+$ and $\wedge$ for $\cdot$, but it's easier to use these equivalent definitions:

(1) An *ideal* of L is a nonempty downset closed under joins.

(2) An ideal I is *principal* if $I = (a]$, i.e., $\{x \in L | x \leq a\}$, for some $a \in I$.

(3) An ideal I is *proper* if $I < L$.

(4) An ideal I is *prime* if I is proper and $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

(5) An ideal I of L is *maximal* if I is proper and there is no ideal J with $I < J < L$.

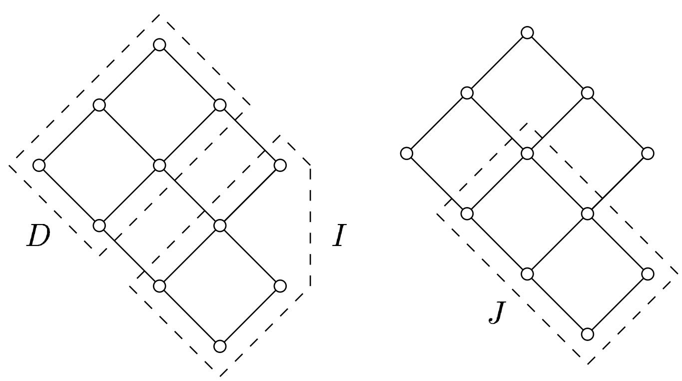

Figure 1: Ideals and a dual ideal

3. Dual ideals (filters) in a lattice L

(1) A *dual ideal* or *filter* in L is a nonempty upset closed under meets.

(2) A dual ideal D is *principal* if $D = [a)$ for some $a \in D$.

(3) A dual ideal D is *proper* if $D < L$.

(4) A dual ideal D is *prime* if D is proper and $x \vee y \in D$ implies $x \in D$ or $y \in D$.

(5) A dual ideal D is *maximal* (an *ultrafilter*) if D is proper and there is no dual ideal E with $D < E < L$.

Note. Often the the terms *filter* and *ultrafilter* are reserved for the case where L is a lattice of subsets, i.e., a sublattice of $\mathrm{Pow}(X)$ for some X.

4. Facts valid in all lattices L

(a) If L is *finite*, every ideal is principal.

(b) The intersection of two ideals is an ideal, and the intersection of any family of ideals is either an ideal or the empty set. It follows that the ideals of L under inclusion form a lattice $\mathrm{Ideals}(L)$, and that if L has a bottom element then $\mathrm{Ideals}(L)$ is complete.

(c) For ideals I and J of L, $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : x \leq i \vee j \text{ for some } i \in I, j \in J\}$. (Contrast with §5(a) below.)

(d) An ideal I is prime if and only if $L \backslash I$ is a dual ideal.

(e) A *principal* ideal is prime if and only if its top element is a meet-prime element.

(f) Prime ideals of L are the same thing as meet-prime elements of $\mathrm{Ideals}(L)$.

5. Facts valid in distributive lattices

(a) $I \vee J = \{i \vee j : i \in I, j \in J\}$. (Contrast with §4(c) above.)

(b) $\mathrm{Ideals}(L)$ is also a distributive lattice.

(c) Any meet-irreducible element of $\mathrm{Ideals}(L)$ is a prime ideal.

(d) *Theorem.* Every ideal I_0 is the intersection of those prime ideals that contain I_0.

Equivalently,

(d′) For an ideal I_0 and $a \notin I_0$, there is a prime ideal I with $I_0 \subseteq I$ but $a \notin I$.

(e) *Corollary.* In a distributive lattice, prime ideals separate points. (In other words, given any two distinct elements, there is a prime ideal that contains one and not the other.)

(f) Any maximal ideal is prime.

6. Facts valid in Boolean lattices

A lattice is *Boolean* if it has a top element 1 and bottom element 0 and every element x has a complement—an element y with $x \wedge y = 0$, $x \vee y = 1$.

Theorem. In a Boolean lattice B, for an ideal I the following are equivalent.

(1) I is a prime ideal;

(2) I separates elements from their complements; in other words, for each $b \in B$, either $b \in I$ or $b' \in I$ but not both;

(3) I is a maximal ideal.

7. Spaces of prime and maximal ideals

For a lattice L, the *prime ideal space* $\Pi(L)$ is the set of all prime ideals of L, with a suitable topology. This is of special interest in the case where L is Boolean, in which case $\Pi(L)$ is a Hausdorff space. However, $\Pi(L)$ is also of interest if L is distributive. In both these cases the lattice can be recovered from the space.

8. Problems

Problem J-1. Let L be the lattice of all finite subsets of an infinite set X. Characterize Ideals(L) up to isomorphism, preferably as a more familiar lattice.

Problem J-2. Prove Fact (e) in §4. If the lattice is distributive, relate this to meet-irreducibility.

Problem J-3. Show that the following conditions are equivalent for an ideal π in a lattice L:

1. π is a prime ideal;
2. $L\backslash\pi$ is a dual ideal;
3. $L\backslash\pi$ is a prime dual ideal; i.e., if $D = L\backslash\pi$, D is a dual ideal such that $x \vee y \in D$ implies $x \in D$ or $y \in D$.

Problem J-4. (a) Explain how homomorphisms of a lattice L onto $\mathbf{2}$ correspond to prime ideals of L.

(b) For $L = \mathbf{2} \times \mathbf{3}$, there are several homomorphisms of L onto $\mathbf{2}$. For each, indicate its kernel by darkening coverings on a copy of L.

Problem J-5. Let X be a countably infinite set. In the Boolean algebra $\mathrm{Pow}_{fin}(X)$ of all finite and cofinite subsets of X, find a prime ideal that is not principal.

Problem J-6. Let $\mathcal{C}$ be the ring of all continuous functions $f : [0,1] \to \mathbf{R}$, where $\mathbf{R}$ is the field of reals and $[0,1] \subseteq \mathbf{R}$. For $x \in [0,1]$, let $M_x = \{f \in \mathcal{C} : f(x) = 0\}$.

(a) Show that M_x is a maximal ideal, for each x.

(b) Show that *every* maximal ideal of $\mathcal{C}$ is of the form M_x.

(c) The *support* of a function f is $\{x : f(x) \neq 0\}$, the set of points where the function does not vanish. Show that the supports of continuous functions on $[0,1]$ form a base for the topology of $[0,1]$. In other words, every open subset of $[0,1]$ is a union of supports.

For a commutative ring R with 1, let $\mathcal{M}$ be its set of maximal ideals. The *support* of an element $r \in R$ is $\{M \in \mathcal{M} : r \notin M\}$. The *maximal ideal space* of R is $\mathcal{M}$ topologized by using the supports of elements as a base.

(d) Show that the maximal ideal space of $\mathcal{C}$ is homeomorphic to $[0,1]$. Thus $[0,1]$ can be reconstructed from $\mathcal{C}$, up to homeomorphism.

(e) Outline how it could be proved that $\mathcal{C}$ is *not* isomorphic to the ring $\mathcal{D}$ of real-valued continuous functions on the unit circle. (It is not necessary to prove the steps.)

Problem J-7. Let L be a lattice and let $I_\gamma, \gamma \in \Gamma$ be a family of ideals of L, possibly infinitely many. Describe (with proof) the elements of the ideal generated by all I_γ, i.e., the smallest ideal containing all the ideals I_γ. Your description should generalize §4(c).