Some higher-level reasoning for rings

Let R be a commutative ring with 1.

1. Prime ideals

Definition. An ideal P of R is *prime* if $P < R$ and $xy \in P$ implies $x \in P$ or $y \in P$.

Proposition. An ideal P of R is prime $\Leftrightarrow$ R/P is an integral domain.

Proof. What does the definition of prime say when expressed mod P?

Corollary. The prime ideals of R are the kernels of homomorphisms of R onto an integral domain.

2. Maximal ideals

Definition. A ideal M of R is *maximal* if $M < R$ and there is no ideal I with $M < I < R$.

Proposition. A ring R is a field $\Leftrightarrow$ R has no ideals except (0) and R.

Proof. Each noninvertible element generates an ideal $< R \ldots$

Proposition. An ideal M of R is maximal $\Leftrightarrow$ R/M is a field.

Proof. By the correspondence theorem, the ideals of R/M correspond to the ideals of R that contain M. M is maximal $\Leftrightarrow$ these are only M and $R \ldots$

Corollary. The maximal ideals of R are the kernels of homomorphisms of R onto a field.

Proposition. Every maximal ideal is prime.

Proof. If M is maximal then R/M is a field, and a field is an integral domain.

3. Ideals of functions

Proposition. Let $\mathcal{R}$ be the ring of all functions from $\mathbf{R} \to \mathbf{R}$ [reals to reals], or the ring of all continuous functions, or all differentiable functions, or all polynomials. For a particular number x, let $M_x = \{f \in \mathcal{R} | f(x) = 0\}$, the ideal of all functions that vanish at x. Then M_x is a maximal ideal of $\mathcal{R}$.

Proof. Consider the evaluation map $e_x : \mathcal{R} \to \mathbf{R}$ given by $e_x(f) = f(x)$. Since e_x is a homomorphism of $\mathcal{R}$ onto $\mathbf{R}$ with M_x as its kernel, M_x is maximal.

4. Existence results

These use Zorn's Lemma (which depends on the Axiom of Choice): In a partially ordered set in which every chain has an upper bound (not necessarily in the chain), there is a maximal element.

Proposition. Every proper ideal I of R can be extended to a maximal ideal.

Proof. Apply Zorn to the partially ordered set $\mathcal{P}$ consisting of the set of proper ideals that contain I. Given a chain, take the union of all its members, which is again an ideal in $\mathcal{P}$. Here it is important that the ring has an identity 1; without that, the union of a chain of proper ideals might not be proper.

Proposition. For any ideal I of R, the intersection of all prime ideals containing I is $\sqrt{I}$ (the *radical* of I, meaning the ideal $\{x \in R : x^n \in I \text{ for some } n\}$).

Proof. Exercise using Zorn.