Well partially ordered sets

1. The concept

Definition. A partially ordered set P is *well partially ordered* (wpo) if

(i) P has the descending chain condition (d.c.c.), i.e., there is no infinite strictly descending chain in P, and

(ii) every antichain in P is finite.

Examples

(a) Any well ordered chain.

(b) ω^n.

(c) The set Ω^* of all finite words (finite sequences) from a finite alphabet Ω, where $\mathbf{s} \leq \mathbf{t}$ means that $\mathbf{s}$ is embedded as a subsequence in $\mathbf{t}$; for example, *aba* is embedded in *cacbac*. (This is a consequence of the Theorem below.)

2. Easily proven observations

(1) P is well partially ordered if and only if every infinite sequence in P has a (weakly) increasing subsequence.

(2) If $P = P_1 \cup P_2$ and P_1, P_2 are well partially ordered, then so is P.

(3) If $P = P_1 \times P_2$ and P_1, P_2 are well partially ordered, then so is P.

(4) If P is well partially ordered and $\phi : P \to Q$ is an isotone map of P onto Q, then Q is well partially ordered.

(5) If P is well partially ordered, then Downsets(P) has d.c.c. Thus any collection of downsets has a minimal member.

3. The main theorem

Theorem. If P is well partially ordered, then so is P^*, the set of finite words (finite sequences) from P, where $p_1 p_2 \cdots p_k \leq q_1 q_2 \cdots q_n$ means that $p_1 \leq q_{i_1}$, $p_2 \leq q_{i_2}$, … , $p_k \leq q_{i_k}$ for some $i_1 < i_2 < \cdots < i_k$.

Note that Example (c) above is the case where P is totally unordered.

Some useful notation: Let us write $p, q, \ldots$ for elements of P and $\mathbf{s}, \mathbf{t}, \ldots$ for elements of P^*. The empty sequence is allowed. Let $P_{\not\geq q, \geq r}$ mean $\{p \in P : p \not\geq q, p \geq r\}$, let $(P^*)_{\geq \mathbf{t}}$ mean $\{\mathbf{s} \in P^* : \mathbf{s} \geq \mathbf{t}\}$, etc. Observe that $P_{\geq q}$ is an upset of P and $P_{\not\geq p}$ is a downset. Also observe that for $q \in P$, $(P_{\not\geq q})^*$ is the same thing as $(P^*)_{\not\geq q}$ (where q is regarded as a one-letter word).

Proof of the Theorem.

Suppose P^* is not wpo. Let D be a downset of P minimal with respect to the property that D^* is not wpo; by Observation 5, such a D exists. Without loss of generality, D is P. Then for all $q \in P$, $(P_{\not\geq q})^*$, or equivalently $(P^*)_{\not\geq q}$, *is* wpo. In contrast, $(P^*)_{\not\geq \mathbf{s}}$ is *not* wpo for some string $\mathbf{s} \in P^*$ in place of q; indeed, let $\mathbf{s}$ be any element of an infinite antichain or strictly decreasing sequence in P^*.

Re-choose $\mathbf{s}$ if necessary to have the least possible length. Write $\mathbf{s} = p_1 p_2 \cdots p_k$. Clearly $\mathbf{s}$ is not empty, so $k \geq 1$. Let $\mathbf{s}'$ be the shorter string $p_1 p_2 \cdots p_{k-1}$ (empty if $k = 1$). Then $(P^*)_{\not\geq \mathbf{s}'}$ *is* wpo.

Now examine $(P^*)_{\not\geq \mathbf{s}, \geq \mathbf{s}'}$. Let $\mathbf{t}$ be one of its elements. Since $\mathbf{t} \geq \mathbf{s}'$, we can write $\mathbf{t} = \mathbf{t}_1 q_1 \mathbf{t}_2 q_2 \cdots \mathbf{t}_{k-1} q_{k-1} \mathbf{t}_k$, where $q_i \geq p_i$. In fact, by choosing $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{k-1}$ as short as possible in turn (so that $\mathbf{t}_1$ stops just before the first element that is $\geq p_1$, etc.), we have $\mathbf{t}_i \not\geq p_i$, for all $i < k$. Since $\mathbf{t} \not\geq \mathbf{s}$, we have $\mathbf{t}_k \not\geq p_k$ as well. In other words, $(P^*)_{\not\geq \mathbf{s}, \geq \mathbf{s}'}$ is an isotone image of part of $(P^*)_{\not\geq p_1} \times P \times (P^*)_{\not\geq p_2} \times P \times \cdots \times P \times (P^*)_{\not\geq p_k}$, so is wpo. But then $(P^*)_{\not\geq \mathbf{s}}$, which is the union of the wpo sets $(P^*)_{\not\geq \mathbf{s}, \geq \mathbf{s}'}$ and $(P^*)_{\not\geq \mathbf{s}'}$, is also wpo, a contradiction.

4. Notes

A. Techniques that were applied to make the proof of the theorem easier to understand:

(a) Use a minimal counterexample in place of an induction, especially if more than one inductive parameter is involved.

(b) Use of "without loss of generality" (wlog) arguments when possible. A single letter can thereby be used in place of two (such as D and P). This approach is an excellent way to analyze a problem in the first place. (The "re-choosing" of $\mathbf{s}$ in the second paragraph is another wlog statement in disguise.)

(c) Split a problem into simpler subproblems, for example by expressing $(P^*)_{\not\geq \mathbf{s}}$ as the union of two wpo sets.

(d) Take technical lemmas useful for the proof and if possible fashion them into more general principles (here, the Observations) that are interesting enough to stand on their own.

(e) Use a suggestive notation, such as $(P^*)_{\geq q}$, etc., that is almost self-explanatory, and choose letters to suggest kinds of objects (elements of P versus sequences).

(f) Use an abbreviation, e.g., "wpo", for a long phrase if the phrase occurs many times. (It is better not to abbreviate a phrase that occurs only several times. In general, do not use abbreviations in statements of theorems if the abbreviations are special to your writeup.)

B. Generalization: The theory is often studied in the more general form of *well quasi-ordered sets*; the results are similar. (A *quasi order* $\leq$ is a relation that is reflexive and transitive but not necessarily antisymmetric, so that $a \leq b$ and $b \leq a$ is allowed for $a \neq b$.)

5. Problems

Problem D-1. Prove Observation (1).

Problem D-2. Prove Observation (2).

Problem D-3. Prove Observation (3).

Problem D-4. Prove Observation (4).

Problem D-5. Prove Observation (5).

Problem D-6. Show the set of all finite trees, as undirected graphs, is a well partially ordered set when partially ordered by embedding.