

## Mal'tsev conditions

### 1. Mal'tsev's Theorem

Two congruences  $\alpha, \beta$  on an algebra  $A$  are said to *permute* if they commute:  $\alpha\beta = \beta\alpha$ .

Here  $\alpha\beta$  is the composition, given by  $x \alpha\beta z$  if and only if there exists  $y$  with  $x \alpha y \beta z$ . Notice that  $\alpha\beta$  and  $\beta\alpha$  are not necessarily equivalence relations.

A variety  $V$  is said to be *congruence permutable* if in any algebra in  $V$  any two congruence relations permute. The famous Russian algebraist Mal'tsev (also transliterated as Mal'cev) gave this characterization:

**1.1 Theorem** (Mal'tsev) For a variety  $V$ , the following are equivalent:

- (a)  $V$  is congruence-permutable;
- (b) there is a term  $p(x, y, z)$  such that in  $V$  these laws hold:

$$\begin{aligned} p(x, x, z) &= z, \\ p(x, z, z) &= x. \end{aligned}$$

For example, the variety of groups satisfies this condition with  $p(x, y, z) = xy^{-1}z$ . The same term, written additively as  $x - y + z$ , then works for rings, since rings have an additive group.

### 2. Mal'tsev conditions

Other theorems of the same general kind have been discovered, where a condition on a variety is characterized using the existence of terms obeying laws of some sort. Such a condition is now called a "Mal'tsev condition". Some typical examples, in addition to congruence permutability, are

- $V$  is *congruence-distributive*. In other words, for any  $\mathcal{A} \in V$ ,  $\text{Con}(\mathcal{A})$  is a distributive lattice.

Example: The variety of all lattices; the variety of all Boolean algebras.

- $V$  is *congruence-modular*. In other words, for any  $\mathcal{A} \in V$ ,  $\text{Con}(\mathcal{A})$  is a modular lattice.

Since the distributive law implies the modular law, any congruence-distributive variety is also congruence-modular. Also, we have:

*Proposition.* Any congruence-permutable variety is congruence-modular.

- $V$  is *arithmetic* (“arithmet’ic”). This means that  $V$  is both congruence-permutable and congruence-distributive.

Example: The variety of rings generated by a finite field.

### 3. Some theorems showing Mal’tsev conditions

In addition to Mal’tsev’s theorem, the following facts hold, among others. Some of them refer to a *majority term*, which means a ternary term  $m(x, y, z)$  obeying the three laws

$$m(x, x, y) = x, m(x, y, x) = x, m(y, x, x) = x$$

in the variety in question.

**3.1 Theorem** (Pixley) For a variety  $V$ , the following are equivalent:

- $V$  is arithmetic;
- there are terms  $p(x, y, z)$  and  $m(x, y, z)$  such that in  $V$ ,  $p$  obeys Mal’tsev’s laws of (b) in Theorem 1.1 and  $m$  is a majority term;
- there is a term  $q(x, y, z)$  such that in  $V$ ,
  - $q(x, x, z) = z$  (minority of entries),
  - $q(x, z, z) = x$  (minority of entries),
  - $q(x, y, x) = x$  (majority of entries).

**3.2 Theorem** (Jónsson) For a variety  $V$ , the following are equivalent:

- $V$  is congruence-distributive;
- for some  $n \geq 2$ , there are terms  $t_0, \dots, t_n$  in  $x, y, z$  such that in  $V$ ,
  - $t_0(x, y, z) = x, t_n(x, y, z) = z$ ;
  - $t_i(x, y, x) = x$ , for all  $i$ ;
  - $t_i(x, x, z) = t_{i+1}(x, x, z)$  for  $i$  even,  $t_i(x, z, z) = t_{i+1}(x, z, z)$  for  $i$  odd.
 (Notice that the case  $n = 2$  is equivalent to the existence of a majority term.)

**3.3 Theorem** (Day, Gumm) For a variety  $V$ , the following are equivalent:

- $V$  is congruence-modular;
- for some  $n \geq 0$ , there are terms  $t_0, \dots, t_n$  and  $p$  in  $x, y, z$  such that in  $V$ ,
  - $t_0(x, y, z) = x$
  - $t_i(x, y, x) = x$ , for all  $i$ ;
  - $t_i(x, z, z) = t_{i+1}(x, z, z)$  for  $i$  even,  $t_i(x, x, z) = t_{i+1}(x, x, z)$  for  $i$  odd.
  - $t_n(x, z, z) = p(x, z, z)$ ,
  - $p(x, x, z) = z$ .

## 4. Problems

**Problem CC-1.** Prove Mal'tsev's theorem.

**Problem CC-2.** Prove Pixley's theorem.

**Problem CC-3.** Another Mal'tsev condition:

(a) Show that the following are equivalent for a variety  $V$ :

(i)  $V$  has a majority term;

(ii) intersections of congruences distribute over composition:

$$\alpha \cap (\beta\gamma) = (\alpha \cap \beta)(\alpha \cap \gamma).$$

(b) Show that a variety with a majority term is congruence-distributive (the case  $n = 2$  of Jónsson's theorem). (Method: Use (ii), generalized to compositions of more than two congruences by an easy induction. Recall that  $\alpha \vee \beta$  is the union of  $\alpha\beta$ ,  $\alpha\beta\alpha$ ,  $\alpha\beta\alpha\beta$ , etc.)