Assignment #8

Due Friday, March 2.

Announcement: No office hour Monday, February 26.

Problem L-1. Ordering integers as usual rather than by division, for a function $f : \omega \to \mathbf{R}$ and $S(n) = \sum_{0 \le k \le n} f(k)$, what does Möbius inversion say, in more familiar terms? (Show your calculation.)

Problem L-2. Find a finite partially ordered set P with bottom element 0 and two other elements r, s with r "covering" s (immediately above s with no other element in between), such that $M(0, s)$ and $M(0, r)$ are both positive.

Problem L-3. A family $\mathcal{F} = \{f_P \in \mathcal{A}(P) | P$ any finite partially ordered set$\}$ is said to be *multiplicative* if if for any P and Q we have $f_{P\times Q}(\langle p_1, q_1\rangle, \langle p_2, q_2\rangle) = f_P(p_1, p_2) f_Q(q_1, q_2)$. Show that the inverses of the functions in a multiplicative family of invertible functions also form a multiplicative family.

(Suggested plan: If the inverse family is g_P, show that $g_P(p_1, p_2) g_Q(q_1, q_2)$ constitutes an inverse of $f_{P\times Q}$ and use the uniqueness of matrix inverses. To start, think what it means to say that $f * g = I$ using a summation; then apply this to the case of $P \times Q$ using pairs; and rewrite this as a double summation, etc.)

(b)Show that Rota's Möbius function $M(x, y)$ is multiplicative, a fact that we have already used to calculate Möbius functions of various partially ordered sets.

Problem L-4. In working with distributions of prime numbers, one approach is to look for functions that have easy apparent approximations and then try to verify those approximations.

(a) Von Mangoldt defined the function Λ on $\{1, 2, \ldots\}$, given by

$\Lambda(n) = \log p$ if $n > 1$ and n is a power of a prime p, or 0 otherwise;

Show that $\Lambda = \mu * \log$, using the number-theoretic version of convolution. (Suggestion: Use inversion.)

(b) Define $\psi(x) = \sum_{n \le x} \Lambda(x)$. Thus ψ picks up a contribution of $\log p$ for each prime power $\le x$, or equivalently, just the log of the highest power of each prime $\le x$. Make a conjecture as to an approximation for ψ, based on visual information, as follows.

Use Mathematica. Start it up, type the command `<< ~baker/psi.m` (shift-return), and then `plotpsi[200]` (shift-return). Then try higher values than 200.

Problem L-5. Heavy-duty calculations with binomial coefficients:

Knuth[1] discusses the following well known formulas, among others. You may assume them without proof when needed. Note that (4) is a generalization of our Problem B-1, which you proved.

(1) $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$, integer $k \neq 0$.

(2) $\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$, integer k, real r.

(3) $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$, integers m, k.

(4) $\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$, integer n, real r.

(5) $\sum_k \binom{r}{k}\binom{s}{n+k} = \binom{r+s}{r+n}$, integer n, integer $r \geq 0$, real s.

The problem: Evaluate $\sum_k \binom{r}{k}\binom{s}{k} k$ (in terms of a single binomial coefficient, with no summation).

(Knuth's suggestion: Use (1) to get rid of the extra k at the cost of a factor s, which can be taken outside the summation. Then use (5) with $n = -1$. Additional examples in Knuth show how to simplify $\sum_{k \geq 0} \binom{n+k}{k}\binom{n}{k}\frac{(-1)^k}{k+1}$ and even more complicated expressions.)

[1] D. E. Knuth, *The Art of Computer Programming*, vol 1, Addison-Wesley, a famous text. Chapter 1 contains a wealth of formulas and problems involving binomial coefficients.