Math 206A **W01 K.**

Assignment #7

Due Friday, February 23. Because of the holiday, you may also submit the paper the following Monday if you wish.

Problem K-1. For the proof that if P is wpo so is P^* , the version in the handout chooses a word $\mathbf{w}$ of minimum length such that $(P^*)_{\not\geq \mathbf{w}}$ is not wpo, while the version in class describes $\mathbf{w}$ as "bad" in the sense that $\mathbf{w}$ participates in an infinite "$\not\leq$" sequence (such as an infinite antichain or strictly descending sequence). Explain why these two descriptions of $\mathbf{w}$ are equivalent.

Problem K-2. Prove directly that if Ω is a finite unordered set then any sequence from Ω^* has an infinite (weakly) increasing subsequence, by imitating the proof of the more general statement for P and P^*, specialized to Ω. (You may still quote any subsidiary facts about wpo's. Where possible, simplify ingredients, such as "downsets of P".)

Problem K-3. (a) Invent and prove a formula, generalizing the inclusion/exclusion formula, for the number of elements of a finite set X having exactly m properties among $P_1, \ldots, P_r$ with $m \leq r$.

(b) Explain how your formula relates to Möbius functions.

Problem K-4. (a) For the digraph G in Figure 1, develop and prove a formula for the number of paths of length n from vertex v_i to vertex v_j $(i, j \in \{1, 2\})$, in terms of a familiar sequence of integers.

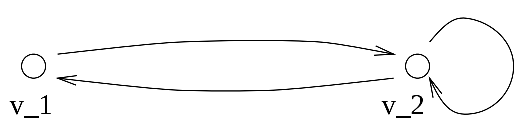

Figure 1: A directed graph

(b) The number of paths of length n from v_1 to v_1 grows approximately geometrically, as r^n for $n \to \infty$. For what value of r? Do this problem in both of two ways: (i) by using (a); (ii) by using eigenvalues. In what sense is the growth approximate in this example?

Problem K-5. (a) For the Euler function $\varphi(n)$ that counts the number of integers relatively prime to k among $1, \ldots, n$, show that $n = \sum_{d|n} \varphi(d)$. (One method: List the fractions $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}$ and reduce them to lowest terms. Another method: A cyclic group of order n has how many elements of each possible order?)

(b) Show that in fact a finite abelian group A is cyclic if and only if A has no more than $\varphi(d)$ elements of each order d with $d|n$.

(c) Use (b) to show that the multiplicative group of a finite field is cyclic. (Tie orders of elements to bounds on the number of roots of polynomials.)

(d) From (a), what formula involving φ arises from Möbius inversion?

(e) Transmute your formula from (d) to get the formula $\varphi(n) = n\prod_{p|n}(1-\frac{1}{p})$, where p denotes primes. (Note: μ is multiplicative : $\mu(mn) = \mu(m)\mu(n)$ if $\gcd(m,n) = 1$.)

Problem K-6. Do the postponed problem on doubly stochastic matrices, provided that we have discussed the discrete version of this problem in class (involving sums of permutation matrices).