Well partially ordered sets

1. The concept

Definition. A partially ordered set P is *well partially ordered* (wpo) if

(i) P has the descending chain condition (d.c.c.), i.e., there is no infinite strictly descending chain in P, and

(ii) every antichain in P is finite.

Examples

(a) Any well ordered chain.

(b) ω^n.

(c) The set Ω^* of all finite "words" (sequences of symbols) from a finite alphabet Ω, where $\mathbf{u} \leq \mathbf{w}$ means that $\mathbf{u}$ is a subsequence of $\mathbf{w}$; for example, *aba* is a subsequence of *cacbac*. (This is a consequence of the Theorem below. *Note:* Ω^* includes the empty word.)

2. Easily proven observations

(1) P is well partially ordered if and only if every infinite sequence in P has a (weakly) increasing subsequence.

(2) If $P = P_1 \cup P_2$ and P_1, P_2 are well partially ordered, then so is P.

(3) If $P = P_1 \times P_2$ and P_1, P_2 are well partially ordered, then so is P.

(4) If P is well partially ordered and $\phi : P \to Q$ is an isotone map of P onto Q, then Q is well partially ordered.

In a partially ordered set P, a *downset* is a subset D such that $x \in D$ and $y \leq x$ imply $y \in D$. The downsets of P form a partially ordered set under inclusion.

(5) If P is well partially ordered, then Downsets(P) has d.c.c. Thus any collection of downsets has a minimal member.

Useful examples of downsets of P associated with each $p \in P$ are

(i) $P_{\leq p} = \{x \in P : x \leq p\}$, the "principal downset" of p.

(ii) $P_{\ngeq p} = \{x \in P : x \ngeq p\}$, the complement of the principal upset $P_{\geq p}$ of p.

3. The main theorem

Theorem. If P is well partially ordered, then so is P^*, the set of finite words from P, where $p_1 p_2 \cdots p_k \le q_1 q_2 \cdots q_n$ means that $p_1 \le q_{i_1}$, $p_2 \le q_{i_2}$, $\ldots$, $p_k \le q_{i_k}$ for some $i_1 < i_2 < \cdots < i_k$.

Note that Example (c) above is the case where P is finite and totally unordered.

Some useful notation: Let us write $p, q, \ldots$ for elements of P and use boldface for words $\mathbf{w} \in P^*$. Let $P_{\not\geq q, \geq r}$ mean $P_{\not\geq q} \cap P_{\geq r}$, etc. Observe that for $q \in P$, $(P^*_{\not\geq q})$ is the same thing as $(P^*)_{\not\geq q}$ (where q is regarded as a one-symbol word).

Proof of the Theorem.

Suppose P^* is not wpo. Let D be a downset of P minimal with respect to the property that D^* is not wpo; by Observation 5, such a D exists. Without loss of generality, D is P. Then for all $q \in P$, $(P_{\not\geq q})^*$, or equivalently $(P^*)_{\not\geq q}$, *is* wpo. In contrast, $(P^*)_{\not\geq \mathbf{w}}$ is *not* wpo for some word $\mathbf{w} \in P^*$ in place of q; indeed, let $\mathbf{w}$ be any element of an infinite antichain or strictly decreasing sequence in P^*.

Re-choose $\mathbf{w}$ if necessary to have the least possible length. Write $\mathbf{w} = p_1 p_2 \cdots p_k$. Clearly $\mathbf{w}$ is not empty, so $k \geq 1$. Let $\mathbf{w}'$ be the shorter word $p_1 p_2 \cdots p_{k-1}$ (empty if $k = 1$). Then $(P^*)_{\not\geq \mathbf{w}'}$ *is* wpo.

Now examine a typical element $\mathbf{u}$ of $(P^*)_{\not\geq \mathbf{w}, \geq \mathbf{w}'}$. Since $\mathbf{u} \geq \mathbf{w}'$, we can write $\mathbf{u} = \mathbf{u}_1 q_1 \mathbf{u}_2 q_2 \cdots \mathbf{u}_{k-1} q_{k-1} \mathbf{u}_k$, where $q_i \geq p_i$. In fact, by choosing $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ as short as possible, we have $\mathbf{u}_i \not\geq p_i$, for all $i < k$. Since $\mathbf{u} \not\geq \mathbf{w}$, we have $\mathbf{u}_k \not\geq p_k$ as well. In other words, $(P^*)_{\not\geq \mathbf{w}, \geq \mathbf{w}'}$ is an isotone image of part of $(P^*)_{\not\geq p_1} \times P \times (P^*)_{\not\geq p_2} \times P \times \cdots \times P \times (P^*)_{\not\geq p_k}$, so is wpo. But then $(P^*)_{\not\geq \mathbf{w}}$, which is the union of the wpo sets $(P^*)_{\not\geq \mathbf{w}, \geq \mathbf{w}'}$ and $(P^*)_{\not\geq \mathbf{w}'}$, is also wpo, a contradiction.

4. Notes

A. The form of the proof: Notice that several familiar pieces of advice were followed to help keep the proof reasonably brief:

(a) Use minimal counterexamples in place of inductions; this enables a single letter to be used in place of subscripted letters, so that the reader has less to think about at one time.

(b) Use "without loss of generality" (wlog) arguments when possible. A single letter can thereby be used in place of two (such as D and P). This approach is an excellent way to analyze a problem in the first place. (The "re-choosing" of $\mathbf{s}$ in the second paragraph is another wlog statement in disguise.)

(c) Split a problem into simpler subproblems, for example in the obtaining of P^* as the union of two wpo sets.

(d) If possible, take technical lemmas useful for the proof and fashion them into more general principles (here, the Observations) that are interesting enough to stand on their own.

(e) Use a suggestive notation, such as $(P^*)_{\geq q}$, etc., that is almost self-explanatory, and choose letters to suggest kinds of objects (elements of P versus words).

(f) Use an abbreviation, e.g., "wpo", for a long phrase if the phrase occurs many times. (It is better not to abbreviate a phrase that occurs only several times. In publications, it is better not to use abbreviations in statements of theorems if the abbreviations are special to your writeup.)

B. Generalization: The theory is often studied in the more general form of *well quasi-ordered sets*; the results are similar. (A *quasi order* $\leq$ is a relation that is reflexive and transitive but not necessarily antisymmetric, so that $a \leq b$ and $b \leq a$ is allowed for $a \neq b$.)