Assignment #6

Due Friday, February 16.

Problem I-1. Recall the theorem of Erdös and Szekeres stating that for each $m \geq 3$ there is a number $N(m)$ such that for $n \geq N(m)$, any n points in the plane in general position include m points forming a convex m-gon. Re-prove this theorem using the following idea[1]: Number the points as $p_1, \ldots, p_n$ and 2-color the triangles they make by saying that $\{p_i, p_j, p_k\}$ with $i < j < k$ has one color if going from p_i to p_j to p_k traverses two sides of that triangle counterclockwise, or the other color if clockwise.

Problem I-2. Recall the other theorem attributed to Erdös and Szekeres, that in a sequence of $n^2 + 1$ distinct integers there is either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$. (a) Re-prove this theorem using Dilworth's theorem and the product ordering $(i, a_i) \leq (j, a_j)$ when $i \leq j$ and $a_i \leq a_j$.

(b) Can you generalize the theorem to the case where the asserted lengths of increasing and decreasing subsequences are possibly different?

Problem I-3. For an n-set X, consider antichains in the Boolean algebra $\mathrm{Pow}(X)$ of subsets of X, partially ordered by inclusion. One way to make an antichain is to take all subsets of some fixed size k, so the antichain has $\binom{n}{k}$ elements. The largest such antichain is obtained for the middle binomial coefficient if n is even, or the middle middle two, if n is odd, so the maximum length of a "horizontal antichain" is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Here $\lfloor\ \rfloor$ is the "floor" (round-down) function.

Could there be a longer antichain formed from subsets of different sizes? "Sperner's Theorem[2]" says no. Prove Sperner's Theorem. (Suggested method: Let $A_1, \ldots, A_m$ be an antichain and let $n_i = |A_i|$. In the set $\mathcal{C}$ of all maximal chains of $\mathrm{Pow}(X)$, collect together all those that go through an A_i in common. Count these, compared to $|\mathcal{C}|$. Develop a relation with $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.)

Problem I-4. In a graph, the *degree* of a vertex is the number of edges from it. A graph is said to be *regular* if all its vertices have the same degree.

[1]This proof was supposedly invented on an exam by a student who had missed the relevant lecture!

[2]There is also a Sperner's Lemma in topology.

Show that a regular bipartite graph has a "perfect matching"—a matching involving every vertex once.

Problem I-5. In a real vector space, a *convex combination* of vectors is a linear combination in which the coefficient are nonnegative and add up to 1, in other words, a weighted average of vectors. A *doubly stochastic matrix* is a nonnegative matrix in which every row sums to 1 and every column sums to 1. Prove a theorem of G. Birkhoff[3]: Every $n \times n$ stochastic matrix is a convex combination of permutation matrices.

[3] There are two Birkhoffs, father and son. The father, George David Birkhoff, proved the ergodic theorem and was the most famous American mathematician of his time; he proved the ergodic theorem and helped put American mathematics on the map in the first part of the last century. The son, Garrett Birkhoff, was a major contributor to lattice theory, among other things.