Partially ordered sets

1. Definitions

A relation $\leq$ on a set P is a *partial order relation* if

(a)	$x \leq x$	(reflexivity)
(b)	$x \leq y$ and $y \leq x$ imply $x = y$	(antisymmetry)
(c)	$x \leq y$ and $y \leq z$ imply $x \leq z$	(transitivity)

$x \geq y$ means $y \leq x$; $x < y$ means $x \leq y$ and $x \neq y$; $x > y$ means $y < x$.

$\langle P, \leq \rangle$ is a *partially ordered set* (or *poset* or *partly ordered set* or *ordered set*) if $\leq$ is a partial order relation on P. (Generally we just say, "the partially ordered set P".) In the following, P and Q refer to partially ordered sets.

The relation $\leq$ is a *total order relation* on P if also

(d) for all x, y, either $x \leq y$ or $y \leq x$.

In this case, $\langle P, \leq \rangle$ is a *chain* or *totally ordered set* or *linearly ordered set.* (In contrast, if instead no two distinct elements are related, then P is an *antichain.*)

In P, a *covers* b if $a > b$ and there is no c with $a > c > b$.

The *Hasse diagram* of a finite partially ordered set P is a diagram indicating the elements of P by circles or dots, connected by lines that indicate the coverings in P. (No lines are drawn horizontal; a non-horizontal line from b up to a indicates that a covers b.)

A map $f : P \to Q$ is said to be *isotone* if f preserves order: $x \leq y \Rightarrow f(x) \leq f(y)$. It is possible, however, for an isotone map to take two unrelated elements to two related elements, or even to the same element.

A map $f : P \to Q$ is said to be an *isomorphism* if f is one-to-one and onto and both f and its inverse are isotone. In this case, P and Q are *isomorphic.*

Note. The best way to show that two partially ordered sets P, Q are isomorphic is to define maps $f : P \to Q$ and $g : Q \to P$, show that f and g are isotone, and show that f and g are inverse to each other, in the sense that $g(f(p)) = p$ and $f(g(q)) = q$ for all $p \in P, q \in Q$. (It is *not* enough to define f and show that f is isotone, one-to-one, and onto.)

For partially ordered sets P, Q, the *direct product* partial order on the set $P \times Q$ is the coordinatewise ordering: $\langle p, q \rangle \leq \langle p', q' \rangle \Leftrightarrow p \leq p'$ and $q \leq q'$.

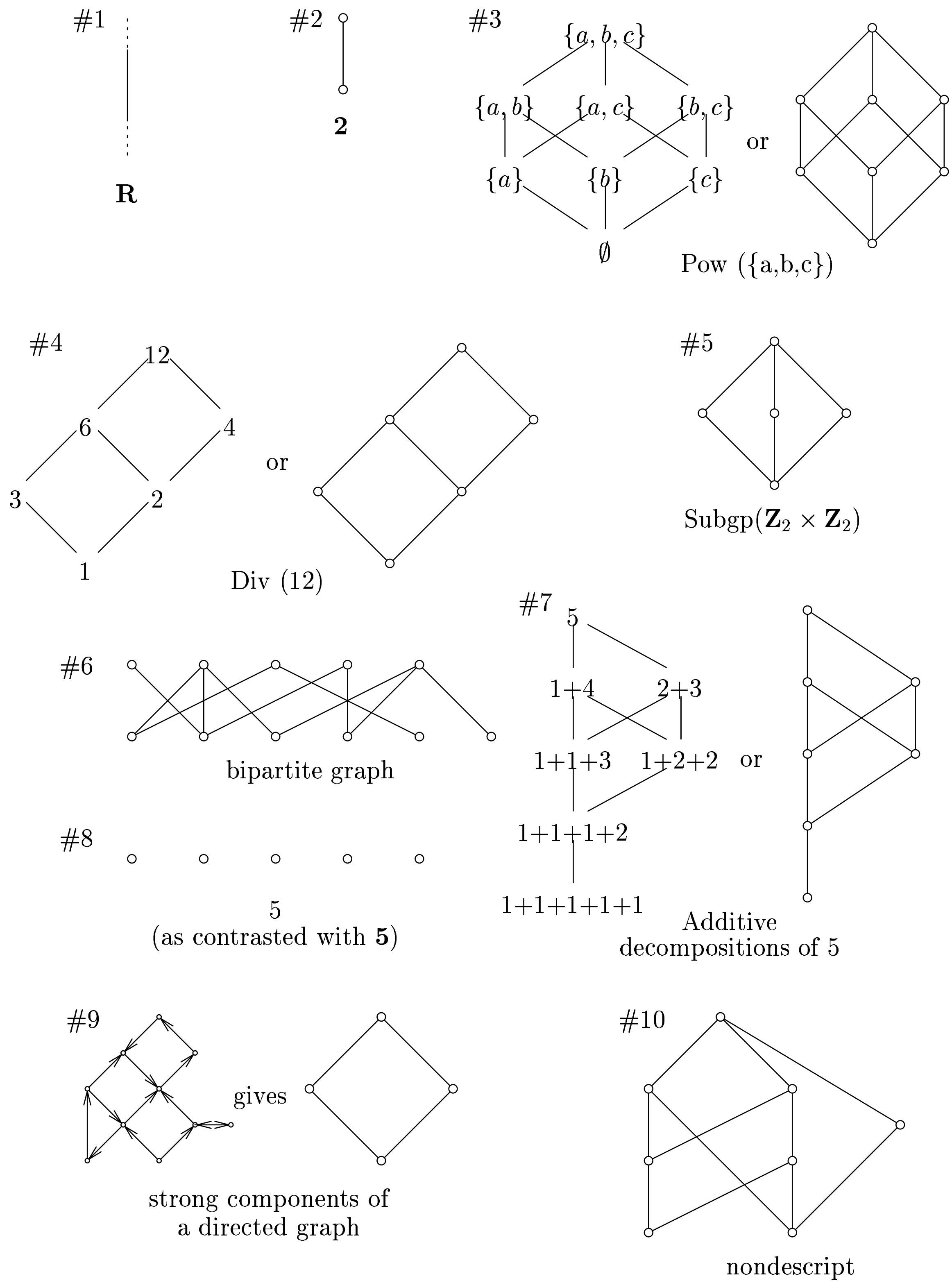

Figure 1: Some examples

The Hasse diagram of $P \times Q$ can be drawn as a copy of Q for each element of P, with P used as a guide for the placement of the copies and for the coverings between them.

A direct product $P_1 \times \cdots \times P_n$ or $\prod_{i \in I} P_i$ is defined similarly.

2. Dilworth's Theorem

2.1 **Definition.** The *width* of a partially ordered set P is the cardinality of the largest antichain. (For example, a chain has width 1.)

Observation. If P is the union of n chains, then P has width at most n.

2.2 **Theorem** (R. P. Dilworth) Let P be a finite partially ordered set of width n. Then P *is* a union of n chains.

This is a kind of minimax theorem, in that it shows that the maximum size of an antichain in P is the minimum number of chains whose union is P. There are a number of combinatorial consequences. Here is one:

Let ρ be a binary relation between finite sets A and B, i.e., $\rho \subseteq A \times B$. A *matching* of A into B is a one-to-one function $f : A \to B$ such that for all $a \in A$, $a \rho f(a)$.

2.3 **Corollary** (P. Hall's matching theorem) Given ρ, a necessary and sufficient condition for the existence of a matching of A into B is that for each $k = 1, 2, \ldots$,

(*) any k elements of A are related to at least k elements of B, in the sense that each of these elements of B is related to at least one of the k elements of A.

(In the proof, the disjoint union of A and B is made into a partially ordered set by declaring $a < b$ when $a \rho b$.)