Sample solutions to Assignment #1

1. An n-element subset of $\{1, \ldots, 2n\}$ is determined by specifying k with $0 \leq k \leq n$, k members of $\{1, \ldots, n\}$ to be in the subset and $n-k$ members of $\{n+1, \ldots, 2n\}$ to be in the subset or equivalently k members of $\{n+1, \ldots, 2n\}$ to be not in the subset. Therefore $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

2. Let a be one of the people. Among the remaining five, either three are friends of a or three are not friends; without loss of generality, suppose the former. Among the three, if any two are friends, those two and a form a trio of mutual friends, as desired. Otherwise, the three are a trio of non-friends.

3. By renaming symbols in each Latin square in the set we can assume that all the squares have first row $1, \ldots, n$. In the position row 2, column 1, each square has an entry among $2, \ldots, n$. If there were more than $n-1$ squares in the set then some two would have the same entry in that position, say with value i. But i against i already occurs in the first row, so orthogonality would be violated. Therefore there are at most $n-1$ squares in the set.

4. (a) A finite sequence of n integers has $n+1$ partial sums (counting the empty sum), so some two partial sums have the same remainder $\pmod{n}$; the difference of these two partial sums is the sum of a run of consecutive terms and is divisible by n.

(b) Suppose person p has shaken hands n_p times. Let $E = \{p \in P : n_p \text{ is even}\}$, $O = \{p \in P : n_p \text{ is odd}\}$. Then $\sum_{p \in E} n_p + \sum_{p \in O} n_p = \sum_{p \in P} n_p = 2H$, where H is the total number of handshakes. Taking remainders mod 2 we get $\sum_{p \in E} 0 + \sum_{p \in O} 1 = 2H \pmod{2} \equiv 0$, i.e., $|O| \equiv 0 \pmod{2}$.

Note. In writing solutions in this course, the goal is to communicate the mathematical idea concisely and clearly, as in a research paper. It is not necessary to write the kind of detailed explanation that you would for teaching undergraduates. Even so, at least mention each crucial point of the logic.

Provide obvious generalizations when you see them. In 4(a), the 13 was supposed to be a temptation to do that! For Problem 1, one person pointed out that by the same argument, $\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$.

It is a stylistic matter in each problem to decide on the use of words versus symbols. Perhaps for 4(a) we could use more symbols and for 4(b), fewer.