Cubic Bézier curves

1. Overview

Bézier curves are a method of designing polynomial curve segments when you want to control their shape in an easy way. Bézier curves make sense for any degree, but we'll concentrate on cubic ones, the most important case. ("Bézier" = "Bay zee ay".)

To specify a cubic Bézier curve, you give four points, called *control points.* The first and last are on the curve; the middle two may not be. When you change the control points, the shape of the curve changes. It is helpful to indicate the control points by connecting them with line segments to form the "control polygon" (although this is not a polygon in the usual sense, as it is not closed). Some examples are shown in Figure 1.

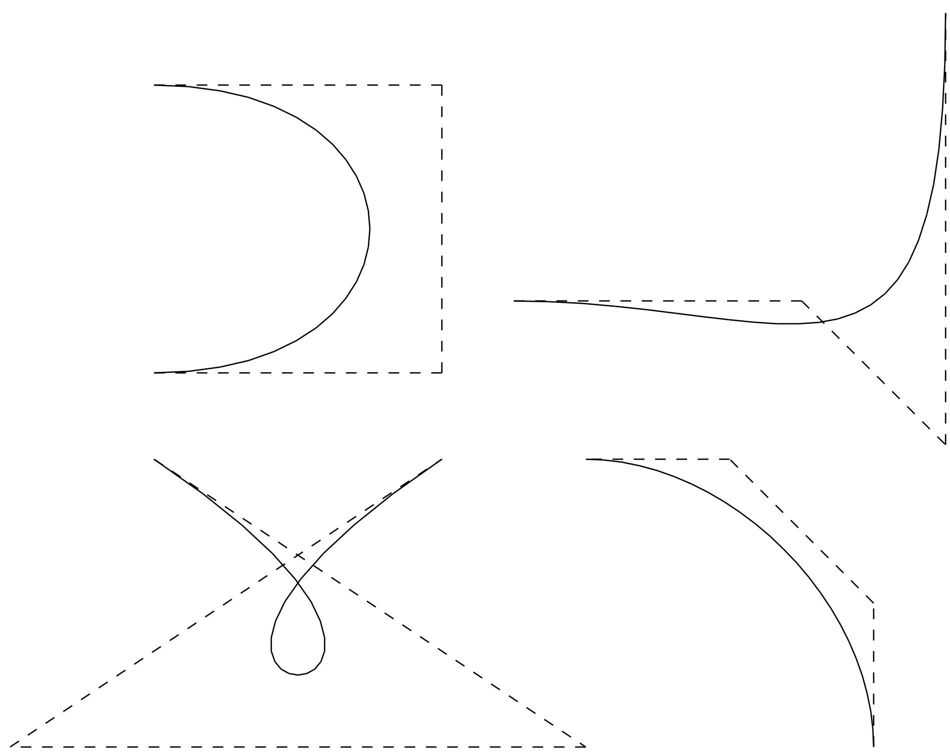

Figure 1: Some Bézier curves

It does not matter which end you consider to be the first and which the last; you get the same points for the curve either way. Observe that the curve is tangent to the first and last "legs" of the control polygon.

2. Details

The Bézier curve is just a particular linear combination of the control points with time-varying coefficients. If the control points are P_0, P_1, P_2, P_3, then the curve is given by

$$P(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3, \text{ for } 0 \le t \le 1.$$

Why these coefficients? They arise in a way related to binomial expansions. Recall that $(s+t)^3 = s^3 + 3s^2t + 3st^2 + t^3$. Now consider these terms individually rather than added together, and put $(1-t)$ for s. You get four functions of t, called the *Bernstein polynomials*[1]. These are

$$\begin{array}{rcccl} J_{3,0}(t) & = & (1-t)^3 & = & 1(1-t)^3t^0 \\ J_{3,1}(t) & = & 3(1-t)^2t & = & 3(1-t)^2t^1 \\ J_{3,2}(t) & = & 3(1-t)t^2 & = & 3(1-t)^1t^2 \\ J_{3,3}(t) & = & t^3 & = & 1(1-t)^0t^3 \end{array}$$

Thus for a Bézier curve,

$$P(t) = J_{3,0}(t)P_0 + J_{3,1}(t)P_1 + J_{3,2}(t)P_2 + J_{3,3}(t)P_3.$$

As you will see, the Bernstein polynomials have nice properties that are reflected in the properties of Bézier curves. Bernstein polynomials and Bézier curves can be defined for any degree n by using the expansion of $(s+t)^n$, but let's continue to concentrate on the case of degree 3, since that case is most frequently used.

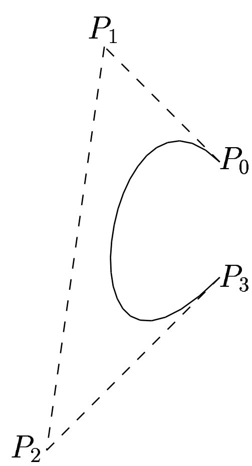

Figure 2: Example

Example. Suppose the control polygon has
$P_0 = (2,3)$, $P_1 = (0,5)$, $P_2 = (-1,-2)$, and $P_3 = (2,1)$, as in Figure 2. Then
$P(t) = J_{3,0}(t)P_0 + J_{3,1}(t)P_1 + J_{3,2}(t)P_2 + J_{3,3}(t)P_3$

[1] The text by Farin and Hansford calls these B_k^n instead of $J_{n,k}$ as here.

$= (1-t)^3(2,3) + 3(1-t)^2 t(0,5) + 3(1-t)t^2(-1,-2) + t^3(2,1)$
$= (2 - 6t + 3t^2 + 3t^3, 3 + 6t - 27t^2 + 19t^3)$

In other words, $P(t) = (x(t), y(t))$ with $x(t) = 2 - 6t + 3t^2 + 3t^3$ and $y(t) = 3 + 6t - 27t^2 + 19t^3$.

3. Some properties of the cubic Bernstein polynomials

- **Values at 0 and 1**:

$$\begin{array}{ll} J_{3,0}(0) = 1 & J_{3,0}(1) = 0 \\ J_{3,1}(0) = 0 & J_{3,1}(1) = 0 \\ J_{3,2}(0) = 0 & J_{3,2}(1) = 0 \\ J_{3,3}(0) = 0 & J_{3,3}(1) = 1 \end{array}$$

- **Unit sum property**: $J_{3,0}(t) + J_{3,1}(t) + J_{3,2}(t) + J_{3,3}(t) = 1$ for all t.

- **Nonnegativity**: $J_{3,k} \geq 0$ for $0 \leq t \leq 1$.

- **Graphs**: See Figure 3.

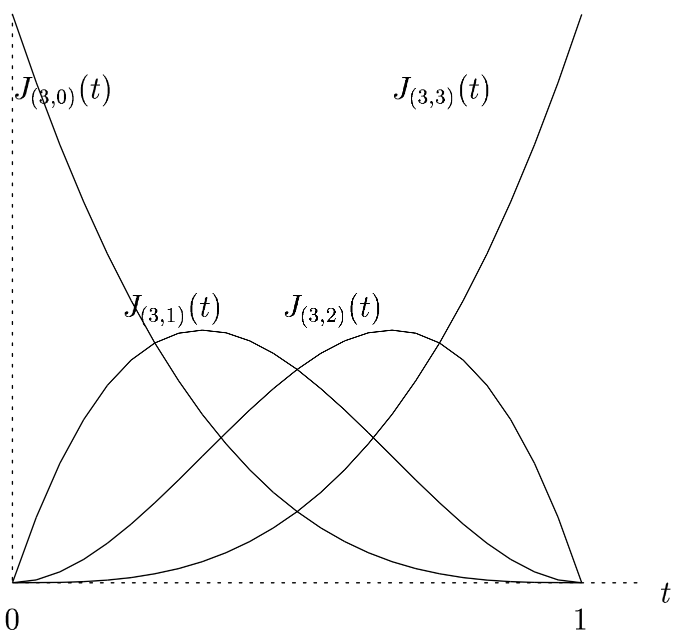

Figure 3: Graphs of Bernstein polynomials of degree 3

- **First derivatives at 0 and 1**:

$$\begin{array}{ll} J'_{3,0}(0) = -3 & J'_{3,0}(1) = 0 \\ J'_{3,1}(0) = 3 & J'_{3,1}(1) = 0 \\ J'_{3,2}(0) = 0 & J'_{3,2}(1) = -3 \\ J'_{3,3}(0) = 0 & J'_{3,3}(1) = 3 \end{array}$$

- **Second derivatives at 0 and 1**:

$$\begin{array}{ll} J''_{3,0}(0) = 6 & J''_{3,0}(1) = 0 \\ J''_{3,1}(0) = -12 & J''_{3,1}(1) = 6 \\ J''_{3,2}(0) = 6 & J''_{3,2}(1) = -12 \\ J''_{3,3}(0) = 0 & J''_{3,3}(1) = 6 \end{array}$$

- **Maximum property**: The maximum value of $J_{3,k}(t)$ occurs at $t = \frac{k}{3}$ for $k = 0, 1, 2, 3$.

- **Linear sum property**: If x_0, x_1, x_2, x_3 are evenly spaced (i.e., form an arithmetic progression), then for all t, $x_0 J_{3,0}(t) + x_1 J_{3,1}(t) + x_2 J_{3,2}(t) + x_3 J_{3,3}(t) = x_0 + t(x_3 - x_0)$, the linear function that runs from x_0 to x_3 as t runs from 0 to 1.

- **Symmetry**: For $k = 0, 1, 2, 3$, $J_{3,k}(1 - t) = J_{3,3-k}(t)$.

- **Basis**: $J_{3,0}(t)$, $J_{3,1}(t)$, $J_{3,2}(t)$, $J_{3,3}(t)$ form a basis for the vector space of all polynomials of degree at most three. (In contrast, the standard basis is $1, t, t^2, t^3$.)

4. Some properties of cubic Bézier curves

Let $P(t)$ be the Bézier curve with four control points P_0, P_1, P_2, P_3.

- **Degree**: $P(t)$ has degree at most 3 (one less than the number of control points).

- **Special values**: $P(0) = P_0$, $P(1) = P_3$.

- **Tangency**: At $t = 0$ the Bézier curve is tangent to the first leg of its control polygon; at $t = 1$ it is tangent to the last leg.

- **Convex hull**: For $0 \leq t \leq 1$, the Bézier curve lies entirely in the convex hull of its control points.

- **First derivative at ends**: $P'(0) = 3(P_1 - P_0)$ and $P'(1) = 3(P_3 - P_2)$.

- **Second derivative at ends**: $P''(0) = 6(P_0 - 2P_1 + P_2)$ and $P''(1) = 6(P_1 - 2P_2 + P_3)$.

- **Affine compatibility**: The Bézier construction is compatible with affine transformations T. In other words, $T(P(t))$ is the same as the point at time t on the Bézier curve with control points $T(P_0)$, $T(P_1)$, $T(P_2)$, $T(P_3)$.

- **Evenly spaced control points**: If P_0, P_1, P_2, P_3 are evenly spaced along a straight line, then $P(t)$ reduces to the usual parametric form of a line segment, namely $P(t) = P_0 + t(P_3 - P_0)$.

- **Maximum influence**: For $k = 0, 1, 2, 3$, the control point P_k has its maximum influence (i.e., its coefficient is at a maximum) at time $t = \frac{k}{3}$.

- **Symmetry**: $P(1 - t)$ is the same as the Bézier curve with control points in the opposite order: P_3, P_2, P_1, P_0.

- **Half-way point**: $P(\frac{1}{2})$ is $\frac{3}{4}$ of the way from the midpoint of the segment $\overline{P_0P_3}$ to the midpoint of the segment $\overline{P_1P_2}$.

- **Generality, algebraically**: For any cubic parametric curve $P(t)$, its portion $0 \leq t \leq 1$ is a Bézier curve, for suitable chosen control points.

- **Generality, geometrically**: Any segment of a cubic parametric curve has the same points as some Bézier curve (with suitably chosen control points).

5. Properties of Bézier curves of any degree

Generalizing the idea of a cubic Bézier curve, a Bézier curve of degree at most n based on $n + 1$ control points $P_0, \ldots, P_n$ is defined by

$P(t) = J_{n,0}(t)P_0 + \cdots + J_{n,n}(t)P_n$,

where the Bernstein polynomial $J_{n,k}(t)$ is defined by $J_{n,k}(t) = \binom{n}{k}(1-t)^{n-k}t^k$.

The properties of cubic Bézier curves, listed above, generalize as you might expect they would. In addition, here is a nice derivative property that explains some of the derivative properties already mentioned for cubic Bézier curves:

- **Derivative at any point**: If $P(t)$ is the Bézier curve with control points $P_0, \ldots, P_n$, then $P'(t) = nQ(t)$, where $Q(t)$ is the Bézier curve of degree at most $n - 1$ based on the control points $\Delta P_0 = P_1 - P_0$, $\Delta P_1 = P_2 - P_1$, $\ldots$,$\Delta P_{n-1} = P_n - P_{n-1}$.

  Thus we get a factor of n, reminiscent of the derivative of x^n, and another Bézier curve based on differences of control points of $P(t)$.

6. Some applications

1. *Making a loop*: Use a Bézier curve in which the first and last control points coincide, as in Figure 4.

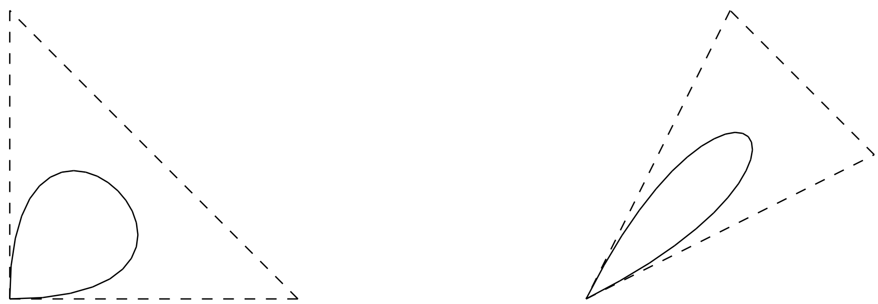

Figure 4: Loops

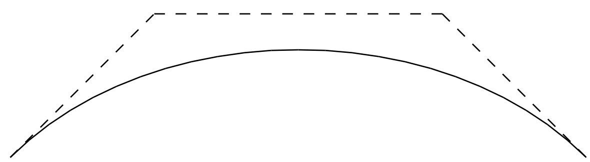

Figure 5: Arcs

2. *Making an arc*: Just use these control points or similar ones, depending on the shape you want, as in Figure 5.

   The control points in this example are $(0,0)$, $(.25,.25)$, $(.75,.25)$, $(1,0)$. If you instead want an arc of this exact shape between other given points Q_0 and Q_1, find an affine matrix that does a rotation, uniform scaling, and translation to move the segment $\overline{(0,0)(1,0)}$ to $\overline{Q_0Q_1}$. To do this, write $Q_0 = (c,d)$ and $(a,b) = Q_1 - Q_0$ and then use the extended matrix $\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ c & d & 1 \end{bmatrix}$.

3. *Making a curved arrow*: Add an arrowhead to a curved arc. For the arc in Figure 5, a suitable arrowhead goes from $(.95,0)$ to $(1,0)$ to $(1,.05)$. For a curved arrow with similar shape in a different position, apply an affine transformation as above. See Figure 6.

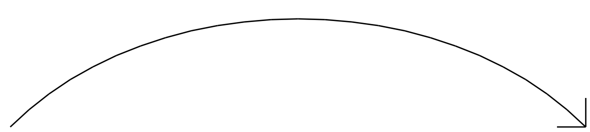

Figure 6: A curved arrow

4. *Making an S-curve*: Just use a control polygon of the kind shown in Figure 7. Adjust as desired. If you add an arrowhead you could use such a curve in a flow chart to show a path from a box on one level to a box on a lower level.

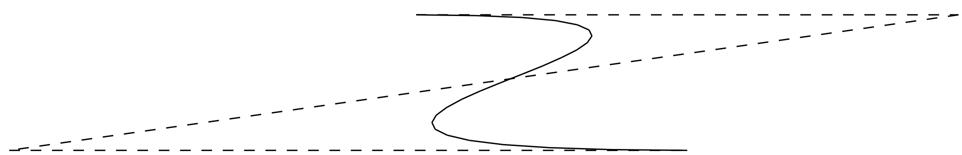

Figure 7: An S-curve

5. *Hermite data*: Suppose you want to find a cubic curve $Q(t)$ that has given values of $Q(0)$, $Q(1)$, $Q'(0)$, and $Q'(1)$. Let's call these values Q_0, Q_1, Q'_0, and Q'_1, respectively. They represent the initial and final positions and velocity vectors, as indicated in the left diagram below. This kind of problem is called an *Hermite* ("air-meet") problem, after the French mathematician Hermite. There is exactly one solution.

   A solution is to use a Bézier curve with appropriate control points P_0, P_1, P_2, P_3. As you see, $P_0 = Q_0$ and $P_3 = Q_1$. Also, from the first derivative property, $3(P_1 - P_0) = Q'_0$ and $3(P_3 - P_2) = Q'_1$. These last two equations can be solved for P_1 and P_2. We get these Bézier control points, as shown in the right-hand portion of Figure 8.

$$\begin{aligned} P_0 &= Q_0 \\ P_1 &= Q_0 + \tfrac{1}{3}Q'_0 \\ P_2 &= Q_1 - \tfrac{1}{3}Q'_1 \\ P_3 &= Q_1 \end{aligned}$$

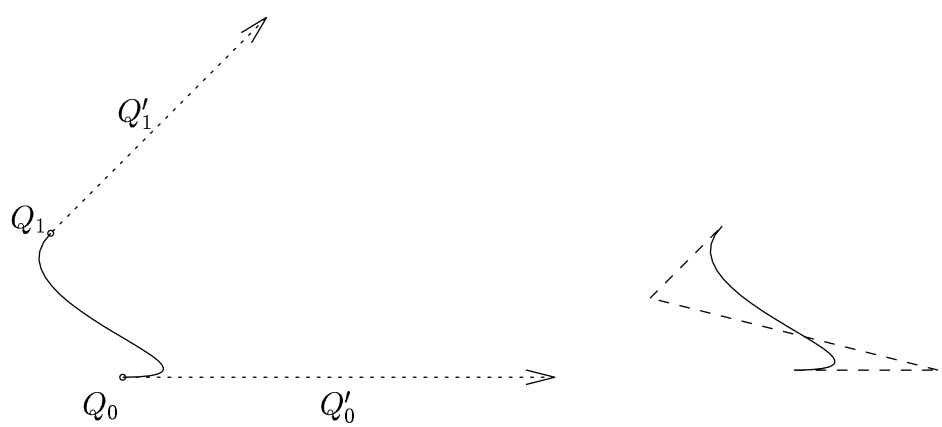

Figure 8: Hermite data

7. Problems

Problem E-1. Prove the half-way-point property of Bézier curves.

Problem E-2. (a) Find the coordinate functions of the Bézier curve with control points (0,0), (1,0), (1,1), (0,2). (b) Sketch this curve. (Instead of plotting many points, it is usually better to plot a few points and find the tangent vectors there to tell the direction of the curve.)

Problem E-3. Here is a graphical construction of a cubic Bézier curve: For each t value you want, first mark the point corresponding to t on each segment $\overline{P_0P_1}$, $\overline{P_1P_2}$, and $\overline{P_2P_3}$. Call the resulting points Q_0, Q_1, Q_2. (For example, if $t = \frac{1}{2}$, each Q_i is the midpoint of its segment.) Then mark the point corresponding to t on $\overline{Q_0Q_1}$ and $\overline{Q_1Q_2}$. Call the resulting points R_0, R_1. Finally mark the point corresponding to t on $\overline{R_0R_1}$. Call the resulting point S_0. Then $P(t) = S_0$.

(a) Carry out this construction for the control points in Problem E-2, for the cases $t = .25$, $t = .5$, $t = .75$.

(b) Prove algebraically that the construction works. (Express the Q_i, R_i and S_0 in terms of t and the P_i, and simplify.)

(c) Prove the Half-way Point property by reasoning graphically—why does the point $P(\frac{1}{2})$ come out three-quarters of the way from the midpoint of $\overline{P_0P_3}$ to the midpoint of $\overline{P_1P_2}$?

Problem E-4. Which of the properties listed for the cubic Bernstein polynomials are fairly evident from the graphs shown, and which are not as evident?

Problem E-5. Prove the unit sum property of the cubic Bernstein polynomials.

Problem E-6. Prove the values given for the first and second derivatives of the cubic Bernstein polynomials at $t = 0$ and $t = 1$.

Problem E-7. Prove the maximum property for the cubic Bernstein polynomials. (Do derivatives help for all k?)

Problem E-8. Prove the linear sum property for the cubic Bernstein polynomials. (One way: First consider just the case where $x_k = \frac{k}{3}$ for $k = 0, 1, 2, 3$. The general case follows from this case and the unit sum property.)

Problem E-9. Prove the basis property for the cubic Bernstein polynomials.

(Since the dimension of the vector space is known to be four, it is enough to show *either* that the cubic Bernstein polynomials span the vector space *or* that they are linearly independent. For the first way: Show how to express each of $1, t, t^2, t^3$ as a linear combination of cubic Bernstein polynomials. For the second way: Given a linear combination that is the zero function, i.e., is zero for all values of t, look at the values of the linear combination at $t = 0$ and at $t = 1$ and also the values of the first derivative at the same places.)

Problem E-10. Give an example of a "cubic" Bézier curve (using the $J_{3,k}$) that actually has (a) degree 1; (b) degree 0; (c) degree 2. (Recall that the *degree* of a polynomial parametric curve is the maximum of the degrees of its coordinate polynomials.)

Problem E-11. For cubic Bézier curves, verify (a) the formulas for the first derivatives at $t = 0$ and $t = 1$; (b) the formulas for the second derivatives at $t = 0$ and $t = 1$; (c) the tangency property (by using (a)). (You may quote any relevant properties of cubic Bernstein polynomials.)

Problem E-12. Prove this version of Taylor's Theorem for cubic Bézier curves: $P(t) = P_0 + 3t(P_1 - P_0) + \frac{t^2}{2}6(P_0 - 2P_1 + P_2) + \frac{t^3}{6}6(-P_0 + 3P_1 - 3P_2 + P_3)$. (One method: algebraic expansion. Another method: Verify that both sides have the same values and first, second, and third derivatives at $t = 0$, because then by the regular Taylor's Theorem they must have the same cubic polynomials for each coordinate.)

Problem E-13. Prove the convex hull property of cubic Bézier curves. (You may quote any needed properties of convex combinations and of cubic Bernstein polynomials.)

Problem E-14. If the four control points for a cubic Bézier curve are all equal, then $P(t)$ is constant. Prove this fact two ways: (a) Using the convex hull property of cubic Bézier curves; and (b) using the unit sum property of Bernstein polynomials.

Problem E-15. Consider the affine compatibility property of cubic Bézier curves. On what property of affine transformations and linear combinations does it depend?

Problem E-16. (a) In pictures of examples like those of Section 1, why is it acceptable not to indicate the x and y axes and a scale? (b) Why is it acceptable not to indicate which end of the curve has $t = 0$? (You may quote relevant properties of Bézier curves, if you explain how they apply.)

Problem E-17. Prove the property of Bézier curves with evenly spaced control points. (You may quote any relevant properties of cubic Bernstein polynomials.)

Problem E-18. Find the coordinate functions of the Bézier curve $P(t)$ for the control points $P_0 = (123, 123)$, $P_1 = (124, 124)$, $P_2 = (125, 125)$, and $P_3 = (126, 126)$. (Do not attempt a long algebraic method; instead, quote any relevant properties of Bézier curves.)

Problem E-19. In the upper left diagram in Figure 1, indicate the point on the curve where each of the two non-end control points has its maximum influence. (Trace the curve and control polygon, regard the vertices as the vertices of the unit square, calculate the two points of maximum influence, and indicate them.)

Problem E-20. For an unspecified positive constant h, consider the loop made by taking a Bézier curve with control points $(0, 0)$, $(h, 0)$, $(0, h)$, $(0, 0)$. (a) Find $P(\frac{1}{2})$. (b) Find the value of t at which the x coordinate is largest, and the corresponding point on the curve. (c) For what value of h would the loop just fit inside a unit square?

Problem E-21. (a) Find control points for a cubic Bézier curve that has exactly the same shape as the arc in Example 2 of Section 5, but goes from $(1, 0)$ to $(2, 1)$. (b) Find points that give an arrowhead so that the whole curved arrow has exactly the same shape as the one in Example 3 of Section 5.

Problem E-22. Prove the formulas given for finding Bézier control points from Hermite data.

Problem E-23. Sketch a cubic parametric curve $Q(t)$ such that $Q(0) = (0, 1)$, $Q(1) = (0, -1)$, $Q'(0) = (6, 0)$, and $Q'(1) = (6, 0)$.

Problem E-24. Prove the algebraic generality property of cubic Bézier curves.

(Method: Write $P(t) = (x(t), y(t))$. You may quote the "basis" property of Bernstein polynomials. A suitable linear combination of them gives the $x(t)$, and similarly for $y(t)$. How should you choose the control points?)

Problem E-25. Prove the geometric generality property of cubic Bézier curves.

(Method: The difference from the algebraic generality property is that you must consider any segment of a cubic parametric curve, not just the segment $0 \leq t \leq 1$. Accordingly, suppose $C(t)$ is a cubic parametric curve and consider a segment given by $a \leq t \leq b$. Say how to make a re-parameterization

$Q(t)$ that gives the same points but for which the same segment is given by $0 \le t \le 1$. Then use the algebraic generality property.)

Problem E-26. (a) Prove this derivative property of Bernstein polynomials: $J'_{n,k}(t) = n\,(J_{n-1,k-1}(t) - J_{n-1,k}(t))$, except that a term is omitted if its second subscript is out of bounds; specifically, $J_{n-1,-1}(t)$ and $J_{n-1,n}(t)$ are omitted.

(Method: You will need the fact that $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Discuss separately the end cases with omitted terms.)

(b) Prove the "derivative property at any point" for Bézier curves of any degree, as given in Section 5.

(c) Use the derivative property at any point to prove the derivative properties listed in Section 4 for cubic Bézier curves.

Problem E-27. (a) Starting from the fact that $(s+t)^2 = s^2 + 2st + t^2$, invent the quadratic Bernstein polynomials and define quadratic Bézier curves (using three control points). (b) Sketch the quadratic Bézier curve with control points $(1,0)$, $(0,0)$, $(0,1)$ for $0 \le t \le 1$. (c) Extend your sketch to $-2 \le t \le 3$.

Problem E-28. Show that in $\mathbf{R}^2$ any quadratic Bézier curve with non-collinear control points can be mapped onto any other quadratic Bézier curve with non-collinear control points by a suitable affine transformation, so that the control polygon of the first is mapped onto the control polygon of the second.

("Noncollinear" means "not in a straight line". You may assume any needed properties of quadratic Bézier curves that are similar to properties of cubic Bézier curves. These curves are parabolas (unless they are lines), so this problem says that arbitrary parabolas can be taken to arbitrary parabolas by an affine transformation, but it says more: This can be done so that the t values match up, with $P(t)$ on one curve going to $Q(t)$ on the other for every t.)

Problem E-29. Find the coordinate functions $x(t)$, $y(t)$, $z(t)$ of a Bézier curve in $\mathbf{R}^3$ with control points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Problem E-30. Show that in $\mathbf{R}^3$, any nonplanar cubic Bézier curve can be mapped by an affine transformation to any other nonplanar Bézier curve. (You may quote any relevant properties of Bézier curves.)

Problem E-31. Show that as a parametric function of t, the first derivative of the cubic Bézier curve with control points $P_0, \ldots, P_3$ is the same as the quadratic Bézier curve with control points $3(P_1 - P_0)$, $3(P_2 - P_1)$, and $3(P_3 - P_2)$. (One method: Expand both sides. Another method: Verify that both sides have the same values and first, second, and third derivatives at $t = 0$, because then by Taylor's Theorem they must have the same cubic polynomials for each coordinate.)

Problem E-32. A certain machine tool is programmed to follow a cubic Bézier curve in $\mathbf{R}^2$. However, it is not necessarily at point $P(t)$ at time t. Instead, the x and y coordinates are driven by separate identical motors and there is a maximum rate at which each motor can go. Therefore the machine is programmed so that at any moment either the x motor or the y motor is going at its maximum speed, whether forwards or backwards. For example, it might be that for the points with $0 \le t \le .21$ on the curve, the x motor is at its maximum rate, then for $.21 \le t \le .53$, the y motor is at its maximum rate, and then for $.53 \le t \le 1.0$ the x motor is at its maximum rate again. In this example, there were three ranges of t. Find the maximum number of ranges there could be for a cubic Bézier curve in general.

(Method: Imagine the path in $\mathbf{R}^2$ of the velocity vector $P'(t)$. By Problem E-31, this curve is a quadratic Bézier curve itself. By an earlier problem, the graph is a parabola. In which regions of $\mathbf{R}^2$ is the x-motor limiting, and in which regions is the y-motor limiting? Into how many pieces could the curve be broken by these regions?)

Problem E-33. For given Hermite data, is there necessarily a quadratic Bézier curve that satisfies the data? (Either give a method or give a counterexample.)

Problem E-34. (a) Invent a "half-way derivative" property for cubic Bézier curves, by expressing $P'(\frac{1}{2})$ in terms of the control points and looking for a geometrical interpretation. (It will involve a vector between midpoints of two sides of the control polygon, times a factor.)

(b) Show that at $t = \frac{1}{2}$ the curve is parallel to the line segment joining the midpoints of the first and last legs of the control polygon.

Problem E-35. When we say a "cubic" Bézier curve we really mean a Bézier curve with four control points and of degree at most 3. Invent a criterion for a cubic Bézier curve to have degree at most two. Express your criterion in terms of the control points. If possible, given a pictorial interpretation.

(Method: A curve is quadratic when its third derivative is always zero. Analogously to the "Second derivative at ends" property, derive a "Third derivative at ends" property. Actually, since the third derivative of a cubic function is constant, the third derivative will be the same at both ends and everywhere else. Now set the third derivative = 0.)

Problem E-36. For each graph in Figure 1, do the following, separately for each: Choose x, y-axes so that the lower-left vertex is at the origin and so that all control points have integer coordinates, with all their coordinates together having greatest common divisor 1. Then calculate $P(\frac{1}{2})$ in decimals, and see if it does seem to match the graph.

Problem E-37. Plot the following functions together on one graph, for $0 \le t \le 1$:
$J_{3,0}(t)$,
$J_{3,0}(t) + J_{3,1}(t)$,
$J_{3,0}(t) + J_{3,1}(t) + J_{3,2}(t)$,
$J_{3,0}(t) + J_{3,1}(t) + J_{3,2}(t) + J_{3,3}(t)$.

Problem E-38. On a single graph, plot the Bézier curve with control points $(1,0)$, $(1,\frac{1}{2})$, $(\frac{1}{2},1)$, $(0,1)$ and also the three other Bézier curves whose control points are the same as these times R_{90°, R_{180°, and R_{270°, respectively. This gives a pretty good approximation to a circle.

Problem E-39. For a Bézier curve of degree 4, with control points $P_0, \ldots, P_4$, invent a Half-way Point Property in terms of P_2 and the midpoints of $\overline{P_0P_4}$ and $\overline{P_1P_3}$.

Problem E-40. Sketch a parametric cubic curve $P(t)$ for which $P(0) = (-1,0)$, $P(1) = (1,0)$, $P'(0) = (-3,3)$, $P'(1) = (3,-3)$.

Problem E-41. Show that for a cubic Bézier curve with control points P_0, P_1, P_2, P_3, the center of mass of the "control triangle" for $P'(t)$ is the same as the "missing leg" $P_3 - P_0$ (as a vector) in the original control polygon. (You may quote the result of Problem E-31.)

Problem E-42. Find the set of t with $0 \le t \le 1$ for which $J_{3,0}(t)$ has the largest value among the four Bernstein polynomials of degree 3. This is the same as the set of t for which P_0 has the largest influence among the four control points. Correspondingly, where does each of P_1, P_2, and P_3 have the largest influence, in terms of sets of t?