Review notes on linear maps

1. Characterizations

Definition. $T : \mathbf{R}^n \to \mathbf{R}^m$ is a *linear map* if

(i) $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$, for any $\mathbf{v}, \mathbf{w}$ ("additivity"), and

(ii) $T(r\mathbf{v}) = rT(\mathbf{v})$, for any vector $\mathbf{v}$ and scalar r ("homogeneity").

Example. Let A be an $m \times n$ matrix, and let T_A be defined by $T_A(\mathbf{x}) = A\mathbf{x}$ (where $\mathbf{x}$ is any column vector). Then T_A is a linear map, by the algebraic properties of matrix operations.

Observations. If T is a linear map then

(a) $T(\mathbf{0}) = \mathbf{0}$ (the origin goes to the origin),

(b) $T(r\mathbf{v}+s\mathbf{w}) = rT(\mathbf{v})+sT(\mathbf{w})$ (T is compatible with linear combinations).

Theorem. *Every* linear map $T : \mathbf{R}^n \to \mathbf{R}^m$ has the form $T_A(\mathbf{x}) = A\mathbf{x}$, for a unique A.

Outline of proof. Use the standard basis of $\mathbf{R}^n$: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \ldots,$

$\mathbf{e}_n = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$. For any T, let A be the matrix whose i-th row is $T(\mathbf{e}^{(i)})$; then $T(\mathbf{x})$ is the same as $A\mathbf{x}$ for $\mathbf{x} =$ one of the $\mathbf{e}^{(i)}$ and so for any $\mathbf{x}$, since every vector $\mathbf{x}$ in $\mathbf{R}^m$ is a linear combination of standard basis vectors and T preserves linear combinations. Thus $T = T_A$.

Remarks

(1) Linear maps leave the origin fixed. We'll also be discussing "affine" maps, which are a generalization in which the origin can move. Because these still take lines to lines, some texts also call affine maps "linear".

(2) If we're thinking about linear maps abstractly we'll use just the letter T; if we have A in mind we'll use T_A. By the Theorem these two points of view are equivalent. Later on we'll use T for other kinds of maps as well.

(3) The definition of $T_A(\mathbf{x})$ as $A\mathbf{x}$ assumes that the n-tuple $\mathbf{x}$ is represented as a row vector; this is a common assumption in computer graphics packages. If we use column vectors, it would be $A\mathbf{x}$. Notes in this course generally assume row vectors.

(4) In projective geometry, it is shown that any one-to-one maps of $\mathbf{R}^n$ onto itself taking lines to lines and the origin to the origin must be a linear map. (This is a deep fact.)

2. Composition of linear maps

Suppose $U : \mathbf{R}^n \to \mathbf{R}^m$ and $T : \mathbf{R}^m \to \mathbf{R}^k$ are linear maps. Then it makes sense to consider the composition $T \circ U$ given by $(T \circ U)(\mathbf{x}) = T(U(\mathbf{x}))$. Notice it's U that is applied first. In terms of the matrices, if $U = T_A$ and $T = T_B$, then $(T \circ U)(\mathbf{x}) = (T_A \circ T_B)(\mathbf{x}) = A(B\mathbf{x}) = (AB)(\mathbf{x}) = T_{AB}(\mathbf{x})$. Thus composition of linear maps corresponds to matrix multiplication. The order of the matrices is the same as the order of the maps when written as a composition.

This explains in particular why matrix multiplication is associative, i.e., why $(AB)C = A(BC)$: Composition is obviously associative, since $(S \circ T) \circ U$ and $S \circ (T \circ U)$ applied to $\mathbf{x}$ are both just $S(T(U(\mathbf{x})))$.

3. Inverses of linear maps

In the preceding section, T and U are *inverses* of each other if $T \circ U$ and $U \circ T$ are both identity functions, i.e., $T(U(\mathbf{x})) = \mathbf{x}$, $U(T(\mathbf{x})) = \mathbf{x}$. If so, write $U = T^{-1}$. This can be the case only if $m = n = k$, so that $T, U : \mathbf{R}^n \to \mathbf{R}^n$.

For the corresponding matrices A and B, this means that they must both be square of the same size and $AB = I$, $BA = I$. In other words, for $T = T_A$ and $U = T_B$, $U = T^{-1}$ if and only if $B = A^{-1}$.

Recall that one practical way to find the inverse of a matrix A is to make a matrix $[A|I]$ and row-reduce it to get $[I|B]$; then $B = A^{-1}$.

4. Determinants

Important: $\det A$ makes sense only if A is *square*! The corresponding linear map T_A therefore is from n-space of some dimension n to itself:
$T_A : \mathbf{R}^n \to \mathbf{R}^n$.

Interpretations of the determinant (assuming $\det A \neq 0$):

If A is 2×2, $|\det A|$ is the factor by which the corresponding linear map changes all *areas*.

If A is 3×3, $|\det A|$ is the factor by which the corresponding linear map changes all *volumes*.

In each case, $\det A > 0$ if and only if the map *preserves orientation.* In the 2×2 case, this means that figures in the domain are not flipped over; in the 3×3 case, this means that a right-hand glove is not turned into a left-hand glove.

Correspondingly, $\det A < 0$ if and only if the map *reverses orientation.* Finally, $\det A = 0$ if and only if the map collapses all areas or volumes to zero, in which case A is *singular.*

Review the mechanics of determinants on your own—both calculations using cofactors and calculations using row-reduction. Unless the matrix you are interested in is very small or has many zero entries, by hand it is fastest to use row-reduction. Except in the 2×2 case, the full expansion with permutations is not usually the best method.

5. Conditions for invertibility.

Proposition. Let A be an $n \times n$ matrix, with $T_A : \mathbf{R}^n \to \mathbf{R}^n$ being the corresponding linear map. Then the following are equivalent:

(1) Every system of linear equations with coefficient matrix A has a unique solution;

(2) A has an inverse;

(3) $\det A \neq 0$;

(4) the columns of A are linearly independent;

(5) the rows of A are linearly independent;

(6) T_A is one-to-one;

(7) T_A is onto.

Definition. In this case, A and T_A are said to be *nonsingular*, or *invertible.*

Notes. (i) This Proposition works over the complex numbers or any other "field" in place of $\mathbf{R}$.

(ii) The word *singular* means "unusual" and should not be confused with the word "single". In particular, (1) says that for a system of linear equations, a *nonsingular* coefficient matrix gives a single solution!

6. Special matrices

(i) The $n \times n$ *identity* matrix $I = \begin{bmatrix} 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 1 \end{bmatrix}$ (other entries 0)

(ii) The $m \times n$ *zero* matrix $\mathbf{O}$ (all entries 0)

(iii) *scalar* matrices $\begin{bmatrix} r & & & \\ & \cdot & & \\ & & \cdot & \\ & & & r \end{bmatrix} = rI$

(iv) *diagonal* matrices in general, $\begin{bmatrix} d_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & d_n \end{bmatrix}$

(v) *rotation* matrices, representing a rigid motion that preserves orientation;

(vi) *reflection* matrices, whose corresponding map gives a reflection in some "mirror" through the origin. In $\mathbf{R}^2$, the mirror will be a line, and in $\mathbf{R}^3$, the mirror will be a plane.

(vii) *shear* matrices, by which we'll mean matrices that are the same as I except that in some row one or more nondiagonal entries can be nonzero. Example: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

7. Problems

(not to be done unless assigned)

Problem C-1. Find a matrix A for which the linear map $T_A(\mathbf{x}) = A\mathbf{x}$ has $T_A \left[\ 1; 0\ \right] = \left[\ 2; 3\ \right]$ and $T_A \left[\ 0; 1\ \right] = \left[\ 1; 4\ \right]$.

Problem C-2. Find a matrix A for which the linear map $T_A(\mathbf{x}) = A\mathbf{x}$ has $T_A(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and $T_A(\begin{bmatrix} 1 \\ 4 \end{bmatrix}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

(Method: Find a matrix B for which the linear map $U(\mathbf{x}) = B\mathbf{x}$ gives $U(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $U(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and a matrix C for which the linear map $V(\mathbf{x}) = C\mathbf{x}$ gives $V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and $V(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$. Then use $A = CB^{-1}$. Why does this work?)

Problem C-3. In $\mathbf{R}^2$, consider the square with vertices $\begin{bmatrix} \pm 1 \\ \pm 1 \end{bmatrix}$. Write down matrices for all linear maps that take this square to itself, including the identity matrix. (There are eight possibilities.)

Problem C-4. For each map on the handout with images of a house, write down the corresponding matrix and its determinant. (Count picture #1 as being the identity map.)

(Method: Look at the images of the standard basis vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; these give the rows of the matrix. If it's not clear which standard basis vector goes to which image vector, then look at how the house and its image are lined up with respect to the standard basis vectors and their images.)