

Interpolation for polynomial parametric curves

1. Parametric curves in general

Let's work with curves in \mathbf{R}^2 , although curves in \mathbf{R}^3 are treated the same. Recall that a curve given parametrically is the same thing as a vector-valued function $P(t)$, i.e., a function $P : \mathbf{R} \rightarrow \mathbf{R}^2$ or a function on just part of \mathbf{R} to \mathbf{R}^2 . We can either write $P(t)$ directly or write the function using coordinates: $P(t) = (x(t), y(t))$.

The curve in \mathbf{R}^2 is really the image of the function, or in other words, the path swept out by the moving point described by $P(t)$, if you think of t as time. Usually, though, we just say "the parametric curve" when we mean the function $P(t)$.

In this course, such functions are usually given one of two ways:

1. By giving the coordinate functions themselves:

Example 1.1 . $P(t) = (\cos t, \sin t)$

Example 1.2 . $P(t) = (t^3, t^2)$ (Figure 1)

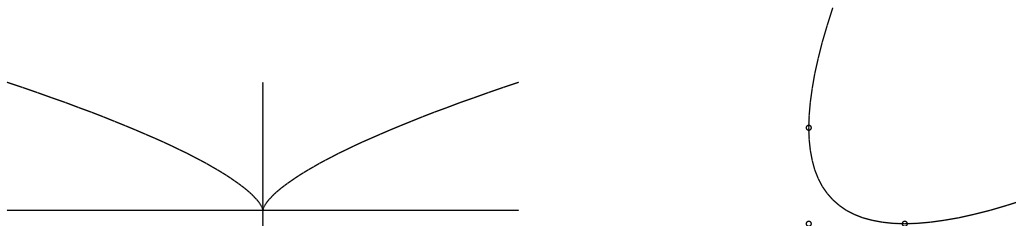


Figure 1: Examples 2 and 4

2. By expressing the curve as a linear combination of given points, where the *coefficients* are functions of t . The curve might go through some of the points or it might not.

Example 1.3 . $P(t) = (1 - t)P_0 + tP_1$ (a line).

Example 1.4 . $P(t) = (t^2 - 2t + 1)P_0 + (2t - 2t^2)P_1 + t^2P_2$ (Figure 1).

An expression such as the one of Example 1.4 may look unfamiliar at first, but notice that for each t , each coefficient is some particular number, so this kind of computation is really nothing new. For instance, in Example 4, $P(\frac{1}{2}) = \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{1}{4}P_2$.

We could call an expression of the second kind a *time-varying linear combination of points*. In graphics the coefficient functions of t are called *blending functions*. For a curve given this way, it is easy to find the coefficient functions:

Observation. For a curve $P(t) = f_0(t)P_0 + \cdots + f_n(t)P_n$, write $P_i = (x_i, y_i)$ and $P(t) = (x(t), y(t))$. Then

$$x(t) = f_0(t)x_0 + \cdots + f_n(t)x_n,$$

$$y(t) = f_0(t)y_0 + \cdots + f_n(t)y_n.$$

Interestingly, $x(t)$ and $y(t)$ can also be regarded as linear combinations of the functions f_i with the numbers x_i or y_i as coefficients, instead of as linear combinations of numbers with functions as coefficients.

In all applications in this course, the coefficient functions will add up to 1 at all times t used. This property is needed to ensure that if the points P_i are translated by some vector \mathbf{b} , then the curve is also translated by \mathbf{b} . (See the Exercises.)

2. Polynomial curves

By a *polynomial curve* in \mathbf{R}^n let us mean simply a parametric curve given by a function $P(t)$ for which each coordinate function is a polynomial. Examples 1.2, 1.3, and 1.4 are polynomial curves. So is this curve:

Example 2.1. $P(t) = (1-t)^3(4, 1) + 3(1-t)^2t(0, -3) + 3(1-t)t^2(0, 3) + t^3(4, -1)$. (See Figure 2.)

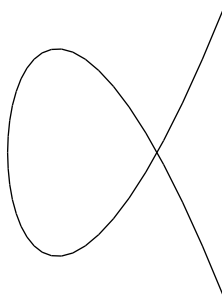


Figure 2: Example 5

The *degree* of a polynomial curve is the maximum of the degrees of the coordination functions. If a polynomial curve $P(t)$ is described as a time-varying linear combination of points, you can see that the degree of $P(t)$ is no larger than the largest degree of the functions $f_i(t)$. (The degree could be less, if powers of t cancel when adding polynomials.)

3. Interpolation

Interpolation for parametric curves means finding a curve $P(t)$ that goes through given data points at given times, as in Figure 3.

In other words, data consists of points P_0, \dots, P_n and times t_0, \dots, t_n . The problem is to find a parametric curve $P(t)$ such that $P(t_k) = P_k$ for each k . As always, let's assume we're working in \mathbf{R}^2 , although everything works similarly in \mathbf{R}^n for any n .

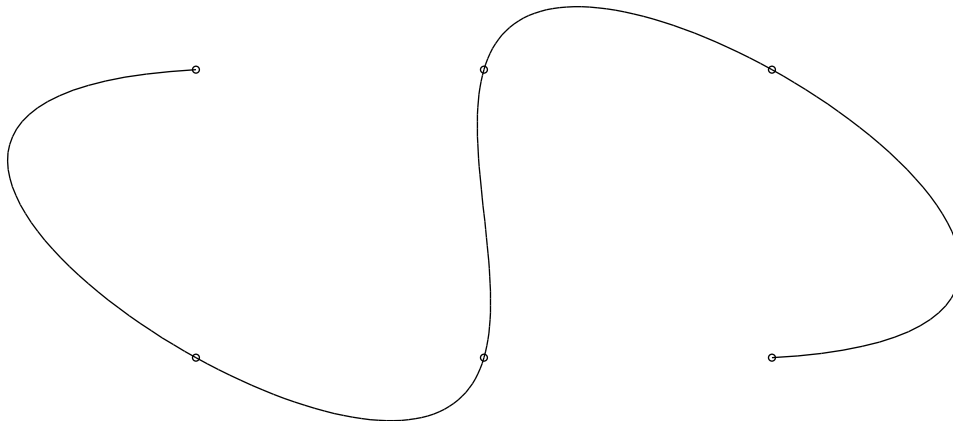


Figure 3: An interpolated curve

But what kind of curve? There are actually several good possibilities, but one of the simplest is to have $P(t)$ be a polynomial curve. What degree should we hope for? Because two points determine a line and three a parabola, it seems reasonable to ask that for $n + 1$ data points P_0, \dots, P_n , the degree of the curve should be at most n . This is always possible:

Theorem. 3.1 (Lagrange). For data points P_0, \dots, P_n and times t_0, \dots, t_n , there is a unique polynomial curve $P(t)$, of degree at most n , such that $P(t_0) = P_0, \dots, P(t_n) = P_n$, provided only that t_0, \dots, t_n are distinct.

A problem fitting this theorem can be called a *Lagrange interpolation* problem.

4. A plan

Because we're starting with given data points, let's look for a solution in the form of a time-varying linear combination

$$P(t) = p_0(t)P_0 + \dots + p_n(t)P_n.$$

Here let's use p_i instead of f_i to emphasize that the functions we seek are polynomials in t . If each p_i is a polynomial in t of degree at most n , then $P(t)$ will be a polynomial curve of degree at most n , as required.

How can we arrange to have $P(t_0) = P_0$? Easily: This happens if $p_0(t_0) = 1$ but $p_1(t_0) = p_2(t_0) = \dots = p_n(t_0) = 0$, because then at time t_0 we get $P(t_0) = 1 \cdot P_0 + 0 \cdot P_1 + \dots + 0 \cdot P_n = P_0$, as desired.

Similarly, we need $p_1(t)$ to have values at t_0, t_1, \dots, t_n of $0, 1, 0, \dots, 0$ respectively. Then we need $p_2(t)$ with a similar property, and so on.

The desired values of $p_0(t), \dots, p_n(t)$, more compactly, are

$p_i(t_i) = 1$ for each i , and

$p_i(t_k) = 0$ for $k \neq i$.

5. Construction of the functions $p_i(t)$

To keep things simple, let's start with the case where $n = 2$ and $t_0 = 3, t_1 = 5, t_2 = 8$, and let's just try to find $p_0(t)$. Thus we want $p_0(t)$ so that $p_0(3) = 1, p_0(5) = 0, p_0(8) = 0$.

As a first goal, let's just try to make a polynomial that is zero at $t = 5$ and at $t = 8$ but not at $t = 3$. That is easy: The function $(t - 5)(t - 8)$ works. It is nonzero at $t = 3$ because the value there is a product of two nonzero factors.

However, at $t = 3$ the value is 10 and not 1 as desired. Fortunately, all we need to do is to divide the function by the 10 and we're done:

$$p_0(t) = \frac{(t - 5)(t - 8)}{10}.$$

To show where this result came from, we could write p_0 as

$$p_0(t) = \frac{(t - 5)(t - 8)}{(3 - 5)(3 - 8)}.$$

Now we're ready for $p_1(t)$. The same reasoning gives

$$p_1(t) = \frac{(t - 3)(t - 8)}{(5 - 3)(5 - 8)}.$$

You can check that p_1 at $t = 3, 5, 8$ has values $0, 1, 0$ respectively. Similarly, $p_2(t) = \frac{(t - 3)(t - 5)}{(8 - 3)(8 - 5)}$.

Now it should be clear how to make $p_0(t) \dots, p_n(t)$ for general n : $p_i(t)$ is a fraction whose numerator is a product of factors $(t - t_j)$ *except for* $j = i$; the denominator is a product of the same factors except with $(t_i - t_j)$ instead of $(t - t_j)$. To write all this more compactly, it is handy to use a sign \prod (capital pi) for a product, the same way that \sum is used for summation.

Theorem 5.1 . The solution to the polynomial interpolation problem can be expressed as

$$P(t) = p_0(t)P_0 + \dots + p_n(t)P_n, \text{ where for each } i,$$

$$p_i(t) = \frac{\prod_{j \neq i} (t - t_j)}{\prod_{j \neq i} (t_i - t_j)} = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}.$$

Proof of Theorem 5.1. Just substitute in $t = t_k$, and you will find that each p_j has the right value, and consequently $P(t_k) = P_k$. Observe that each $p_i(t)$ has degree n . Each coordinate function of $P(t)$ is a linear combination of $p_0(t), \dots, p_n(t)$ and so has degree at most n .

Proof of Theorem 3.1. The existence of the required solution is shown by Theorem 5.1. The uniqueness is shown in the Exercises.

Note. The polynomials $p_i(t)$ depend only on n and the times t_0, \dots, t_n , and not on the data points P_i . In Figure 4 are shown graphs of the polynomials $p_i(t)$ for $n = 4$ and times 0, 1, 2, 3, 4.

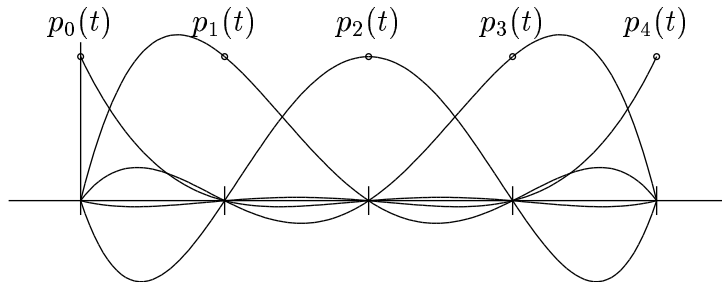


Figure 4: Graphs of basis polynomials

6. Examples

To find a curve of degree at most 2 such that $P(-1) = (1, 0)$, $P(0) = (0, 0)$, and $P(1) = (0, 1)$:

We have $n = 2$, $t_0 = -1$, $t_1 = 0$, $t_2 = 1$.

$$p_0(t) = \frac{(t - 0)(t - 1)}{(-1 - 0)(-1 - 1)} = \frac{1}{2}(t^2 - t)$$

$$p_1(t) = \frac{(t - (-1))(t - 1)}{(0 - (-1))(0 - 1)} = -(t^2 - 1)$$

$$p_2(t) = \frac{(t - (-1))(t - 0)}{(1 - (-1))(1 - 0)} = \frac{1}{2}(t^2 + t)$$

$P(t) = \frac{1}{2}(t^2 - t)(1, 0) - (t^2 - 1)(0, 0) + \frac{1}{2}(t^2 + t)(0, 1) = (\frac{1}{2}(t^2 - t), \frac{1}{2}(t^2 + t))$, as shown in the diagram.

Figure 5 shows this and two other examples.

In the lower example with $n = 10$, observe that the function is so wavy that it doesn't follow the data points very well. This is a disadvantage of working with polynomial functions of higher degrees. Moreover, by the

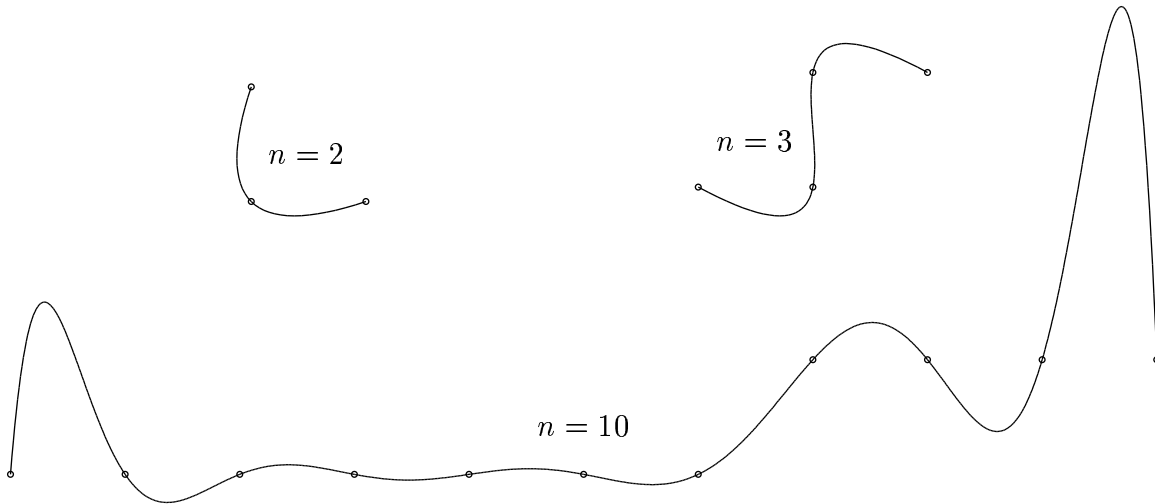


Figure 5: Additional examples

uniqueness property (Theorem 3.1) there is no other polynomial curve that has degree at most 10 and fits the data points.

7. Newton's approach to interpolation

There is also an approach due to Newton for solving polynomial interpolation problems by using "divided differences". The method will be explained here for the case $n = 3$, without proof, but it works the same for every n .

Suppose, then, that times t_0, \dots, t_3 are given along with data points P_0, \dots, P_3 . As usual, we want a polynomial curve of degree at most 3 with $P(t_i) = P_i$ for each i .

Step 1. Compute quantities

$$P_{01} = \frac{P_1 - P_0}{t_1 - t_0}, \quad P_{12} = \frac{P_2 - P_1}{t_2 - t_1}, \quad P_{23} = \frac{P_3 - P_2}{t_3 - t_2}.$$

Step 2. Compute quantities

$$P_{012} = \frac{P_{12} - P_{01}}{t_2 - t_0}, \quad P_{123} = \frac{P_{23} - P_{12}}{t_3 - t_1}.$$

Step 3. Compute the quantity

$$P_{0123} = \frac{P_{123} - P_{012}}{t_3 - t_0}.$$

Step 4.

Let $P(t) = P_0 + (t - t_0)P_{01} + (t - t_0)(t - t_1)P_{012} + (t - t_0)(t - t_1)(t - t_2)P_{0123}$.

(For general n there would be steps 1 through $n + 1$.)

The quantities $P_{i\dots j}$ are called *divided differences*. They can be thought of as making a triangular table, as shown. A column of t -values has been added for convenience. The boxes indicate the entries actually used in the last step. (The other entries aren't wasted, as they have been used in computing the boxed entries.)

t	P_i	$P_{i,i+1}$	$P_{i,i+1,i+2}$	$P_{i,i+1,i+2,i+3}$
t_0	P_0			
		P_{01}		
t_1	P_1		P_{012}	
		P_{12}		P_{0123}
t_2	P_2		P_{123}	
		P_{23}		
t_3	P_3			

As you see, each entry $P_{i,\dots,j}$ with more than one subscript is obtained as a quotient, where the numerator is the difference of the entries immediately to the left ($P_{i+1,\dots,j} - P_{i,\dots,j-1}$), and the denominator is the difference of the t -values for the outer subscripts ($t_j - t_i$).

This method is often easier to do by hand than the blending-function method, and it is also easier to program, but it is less helpful in understanding other computer graphics methods for curves.

Example: The second curve in Figure 5 has data points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(2, 1)$. The table of divided differences is as follows. (The denominators of $\frac{1}{2}$ come from $2 - 0$ and $3 - 1$ and the $\frac{1}{3}$ comes from $3 - 0$.)

t	P_i	$P_{i,i+1}$	$P_{i,i+1,i+2}$	$P_{i,i+1,i+2,i+3}$
0	$(0, 0)$			
		$(1, 0)$		
1	$(1, 0)$		$(-\frac{1}{2}, \frac{1}{2})$	
		$(0, 1)$		$(\frac{1}{3}, -\frac{1}{3})$
2	$(1, 1)$		$(\frac{1}{2}, -\frac{1}{2})$	
		$(1, 0)$		
3	$(2, 1)$			

Thus the curve is $P(t) = (0, 0) + t(1, 0) + t(t-1)(-\frac{1}{2}, \frac{1}{2}) + t(t-1)(t-2)(\frac{1}{3}, -\frac{1}{3})$.

8. Some useful properties of polynomials

Except as noted, it will be assumed that polynomials have real numbers as coefficients, rather than any complex numbers. Some of these properties will be useful for the Exercises; others are listed for the sake of completeness.

- **Factor corresponding to a root.** If $f(t)$ is a polynomial and a is a root of $f(t)$, then $(t - a)$ is a factor of f .

(The reason: You can always do a long division of $f(t)$ by $(t - a)$ to get a quotient $q(t)$ and a remainder $r(t)$, which being of lower degree than $(t - a)$ is simply a constant r . Thus $f(t) = q(t)(t - a) + r$. Put $t = a$; since $f(a) = 0$ you get $0 = 0 + r$, so $r = 0$. In other words, $f(t) = q(t)(t - a)$, so $(t - a)$ is a factor of f .)

- **Number of roots.** A nonzero polynomial of degree n has at most n distinct roots.

(The reason: Not more than n different terms $(t - a_i)$ could be factors of $f(t)$, since $f(t)$ is of degree n . Actually, this reason requires further explanation: When you have factored out some of the terms $(t - a_i)$ for roots a_i of $f(t)$ and you have an equation $f(t) = (t - a_1)(t - a_2) \dots (t - a_k)q(t)$, you need to observe that the remaining roots of $f(t)$ are also roots of $q(t)$, so you can keep factoring.)

- **Expression from function.** If f and g are polynomials that are equal as real functions, then they have the same coefficients.

(In other words, in talking about “equal polynomials” we don’t have to say whether we mean equal as functions or equal as polynomial expressions. By way of contrast, in modern algebra one studies polynomials with coefficients in finite fields; in this setting two polynomials with different coefficients can give the same function.)

- **Agreement in many places gives agreement everywhere.** If two polynomials $f(t)$ and $g(t)$ of degree at most n agree at $n + 1$ or more values of t then $f = g$, i.e., $f(t) = g(t)$ for all t .

(The reason: The values of t at which f and g agree are roots of the difference $f(t) - g(t)$. If this difference is nonzero it can have at most n roots.)

- **Continuation property.** If f and g are polynomials that agree on an interval, then they are the same polynomial.

(For example, if $f(t) = g(t)$ for $0 \leq t \leq 0.000001$, then $f(t) = g(t)$ for *all* values of t .)

- **Unique interpolation property (Lagrange property).** There is exactly one polynomial $f(t)$ of degree at most n that has given values at $n + 1$ distinct values of t (i.e., $f(t_0) = y_0, \dots, f(t_n) = y_n$).

(This is just a nonparametric version of Theorem 3.1. The uniqueness comes from the agreement property above.)

- **Unboundedness:** A nonconstant polynomial is unbounded. In other words, if $f(t)$ is nonconstant there is no constant B such that $|f(t)| \leq B$ for all t .

(The reason: If $f(t) = c_n t^n + \dots + c_1 t + c_0$, with $c_n \neq 0$, for large values of t the leading term $c_n t^n$ overwhelms the other terms and makes the value of $f(t)$ large. More formally, if $|c_n t^n + \dots + c_1 t + c_0| \leq B$ for all t with $c_n \neq 0$, consider values of t with $t > 0$ and divide through by t to get $|c_n + \frac{c_1}{t} + \dots + \frac{c_0}{t^n}| \leq \frac{B}{t}$. Then let $t \rightarrow \infty$. You get $|c_n| \leq 0$, a contradiction.)

- **Existence of real roots for odd degree.** Any polynomial of odd degree with real numbers as coefficients must have at least one real root.

(The reason: Suppose the polynomial is $f(x)$ and its leading coefficient is positive. For a large negative value of t , $f(t) < 0$; for a large positive value of t , $f(t) > 0$. By continuity, in between somewhere there is a t with $f(t) = 0$. If the leading coefficient is negative, similar reasoning applies.)

- **Existence of complex roots.** Any nonconstant polynomial, with real coefficients or with even complex coefficients, has at least one complex root.

This is one statement of the *Fundamental Theorem of Algebra*. The following fact is another, equivalent statement:

- **Complex factorization.** If complex numbers are used, any polynomial can be factored completely into linear factors:

$$f(t) = c(t - a_1)(t - a_2) \dots (t - a_n).$$

- **Complex roots of real polynomials.** If a real polynomial is factored into complex linear factors, the non-real roots occur in conjugate pairs. Thus if there is one factor $(t - (3 + 2i))$, then there must be one factor $(t - (3 - 2i))$.

9. Problems

Problem X-1. Write Examples 1.3 and 1.4 in coordinate form. (For the coordinates of the given points, just write $P_i = (x_i, y_i)$.)

Problem X-2. For the (non-polynomial) parametric curve $P(t) = (\cos t, \sin t)$, sketch the curve, the graph of x against t , and the graph of y against t , for $-\pi \leq t \leq 3\pi$.

Problem X-3. Find $P(t)$ explicitly in the upper right example of Figure 5, by using basis functions. Assume that the data points are $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (1, 1)$, and $P_3 = (2, 1)$, with $t_i = i$ for $i = 0, 1, 2, 3$.

Problem X-4. Find a quadratic polynomial curve $P(t)$ (or really, a polynomial curve $P(t)$ of degree *at most two*) such that $P(-1) = (1, 5)$, $P(0) = (2, 1)$, and $P(1) = (3, 5)$, by using basis functions. Give explicit coordinate functions $x(t)$, $y(t)$.

Problem X-5. Find a cubic polynomial curve $P(t)$ (or really, a polynomial curve $P(t)$ of degree *at most three*) such that $P(0) = (1, 0)$, $P(1) = (2, 1)$, $P(2) = (9, 2)$, and $P(3) = (28, 3)$. Use basis functions.

Problem X-6. Solve Problem X-5 using divided differences.

Problem X-7. Suppose that you want to make a parametric curve $P(t)$, polynomial or not, to interpolate data points P_i with $P(t_i) = P_i$ for $i = 0, \dots, n$, and suppose that you plan to express $P(t)$ as a linear combination with time-varying coefficients. What conditions on the blending functions (the coefficient functions) are sufficient to guarantee that $P(t)$ does interpolate as required?

Problem X-8. Find the Lagrange blending functions of degree one.

Problem X-9. (a) Find a quadratic parametric curve that crosses the unit circle $x^2 + y^2 = 1$ in four points. (b) Find a cubic parametric curve that crosses the unit circle in six points. (In both parts, the equations should be

explicit but a graphical proof is sufficient. One method for (b): If you can find a suitable graph $y = ax^3 + bx^2 + cx + d$, you can express the same graph in parametric form as $(t, at^3 + \dots + d)$.

Problem X-10. Show that the unit circle $x^2 + y^2 = 1$ cannot be represented as a polynomial parametric curve. In fact, show that a polynomial parametric curve of degree $n > 0$ cannot touch a unit circle in more than $2n$ distinct points, much less follow the circle all the way around. For example, a parabola cannot touch a circle in more than four points.

(Method: Suppose the curve is $P(t) = (x(t), y(t))$ of degree n . For any t for which $P(t)$ is on the circle, we get $x(t)^2 + y(t)^2 = 1$. Apply the agreement property of polynomials from §8 to show that if more than $2n$ points of the curve are on the circle, then *all* points of the curve are on the circle. Apply the unboundedness property from §8 to show that this is impossible. Then you have made a proof by contradiction.)

Problem X-11. (a) In contrast to the preceding problem, show that the following method does give a parametrization of the unit circle (except for one point) by rational functions (ratios of polynomials): Calculate the point (x, y) where the line of slope m through the point $(-1, 0)$ on the circle intersects the circle a second time. Your answer will have x and y as rational functions of m , say $x(m)$ and $y(m)$. Now write t for m , to use a more familiar parameter. Check that $x(t)^2 + y(t)^2$ is always 1.

(b) Which point of the circle is missing?

(c) Try some rational-number values for t ; the corresponding x and y should lead to some “Pythagorean triples” such as 3, 4, 5 with $3^2 + 4^2 = 5^2$ and 5, 12, 13 with $5^2 + 12^2 = 13^2$. Reduce each triple you get to lowest terms, e.g., use 3, 4, 5 and not 6, 8, 10. Find three triples in lowest terms other than the ones just mentioned.

Problem X-12. Suppose that you have chosen a list $f_0(t), \dots, f_n(t)$ of blending functions, not necessarily polynomials, for which $f_0(t) + \dots + f_n(t) = 1$ for all t with $a \leq t \leq b$. Then given a list of points P_0, \dots, P_n you can make a curve by using the functions $f_i(t)$ as coefficients of a linear combination $P(t) = f_0(t)P_0 + \dots + f_n(t)P_n$. Explain how you know that this process is compatible with affine transformations. In other words, show that if T is an affine transformation and $Q_i = T(P_i)$ for each i , then at all times t with $a \leq t \leq b$ you get $T(P(t)) = Q(t)$, where $Q(t)$ is the curve made with the same blending functions but based on the points Q_i . (Method: What property of affine transformations and linear combinations is relevant?)

Problem X-13. A useful observation is that a polynomial, say $1 + 2t + 7t^2 - 5t^3$, can be written as a matrix product $\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \\ -5 \end{bmatrix}$.

Similarly, the polynomial curve $(1 + 2t + 7t^2 - 5t^3, 4 - t + t^2)$ can be written as $\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 7 & 1 \\ -5 & 0 \end{bmatrix}$.

As you see, the columns of the right-hand matrix represent the coefficients of the x and y coordinate polynomials. (It is usually best to write polynomials in this situation with increasing powers of t , although some texts do not.)

Express Example 1.2 in matrix form.

Problem X-14. Another use of matrices is to make a “matrix of points”, a matrix in which each row represents a point. For example, if data points

P_0, \dots, P_n are given, the corresponding matrix is $P_* = \begin{bmatrix} P_0 \\ \dots \\ P_n \end{bmatrix}$. Then a

linear combination of these points with blending functions f_0, \dots, f_n can be expressed by the matrix product $P(t) = \begin{bmatrix} f_0(t) & \dots & f_n(t) \end{bmatrix} P_*$.

(a) Express Example 1.4 in this form, for the case $P_0 = (7, 2)$, $P_1 = (-1, 4)$, $P_2 = (3, 6)$.

(b) Combine (a) with the method of the preceding problem to write the same example in the form $P(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} MP_*$, where M is a certain matrix of numbers. (Do not multiply out.)

Problem X-15. Complete the details in the proof of Theorem 5.1.

Problem X-16. A *Van der Monde* matrix is a matrix of the form

$$V = \begin{bmatrix} 1 & t_0 & t_0^2 & \dots & t_0^n \\ 1 & t_1 & t_1^2 & \dots & t_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & t_n & t_n^2 & \dots & t_n^n \end{bmatrix}$$

This matrix is $(n + 1) \times (n + 1)$. The t_i can be any numbers. For example,

$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & \pi & \pi^2 \end{bmatrix}$ is a van der Monde matrix.

A theorem is that the determinant of a van der Monde matrix is the product of the $\frac{n(n+1)}{2}$ terms $(t_j - t_i)$ with $i < j$, or symbolically, $\prod_{i < j} (t_j - t_i)$. For example, for a 3×3 van der Monde matrix, the determinant is $(t_1 - t_0)(t_2 - t_0)(t_2 - t_1)$.

- (a) Verify this theorem in the case $n = 2$.
 - (b) Explain why a van der Monde matrix must be nonsingular if the numbers t_0, \dots, t_n are distinct.
 - (c) Find the volume of the tetrahedron with vertices $(8, 8^2, 8^3)$, $(9, 9^2, 9^3)$, $(10, 10^2, 10^3)$, $(11, 11^2, 11^3)$, using only arithmetic that is so easy you could do it in your head.
- (Method: Recall that the volume can be expressed as $\frac{1}{6}$ of the absolute value of a certain determinant whose last column consists of 1's. Changing the order of the columns does not change the absolute value of a determinant.)

Problem X-17. Here is a different approach to solving a Lagrange interpolation problem. As usual, let points P_i in \mathbf{R}^2 and distinct times t_i be given, for $i = 0, \dots, n$. According to a preceding problem, a solution can be written in the form $P(t) = \begin{bmatrix} 1 & t & t^2 & \dots & t^n \end{bmatrix} C$, where C is some $(n+1) \times 2$ matrix of coefficients. The interpolation conditions $P(t_i) = P_i$ can then be written as $\begin{bmatrix} 1 & t_i & t_i^2 & \dots & t_i^n \end{bmatrix} C = P_i$. These conditions can be combined into a single matrix equation $VC = P_*$, where V is a van der Monde matrix.

- (a) Explain why there must be a solution for the matrix C of coefficients and why this solution is unique. (You may quote any needed properties of van der Monde matrices.)
- (b) Explain how you could solve Lagrange problems on a computer in this way if a row-reduction routine were available.

Problem X-18. Prove the uniqueness part of Theorem 5.1: There is only one polynomial curve $P(t)$ of degree at most n for which $P(t_0) = P_0, \dots, P(t_n) = P_n$, where the P_i and t_i are any given points and distinct times.

(Method: Apply the uniqueness property of §8 to the coordinate polynomials.)

Problem X-19. Using the “Agreement in many places” property in §8, explain how you get the (a) the “Expression from function” property and (b) the “Continuation” property.

Problem X-20. Find an example of a polynomial curve $P(t)$ that goes through data points that are in a rectangular window, such that the curve wanders out of the window between two of the data points.

(You should give explicitly the bounds of the window, the coordinate functions of the curve, and the time at which the curve is outside the window. Such an example will contrast with other kinds of polynomial curves to be studied soon.)

Problem X-21. Consider the Lagrange blending functions $p_0(t), \dots, p_n(t)$ for given distinct t_0, \dots, t_n .

(a) Show that for all t , $p_0(t) + \dots + p_n(t) = 1$.

(b) Show that for all t , $t_0 p_0(t) + t_1 p_1(t) + \dots + t_n p_n(t) = t$.

(Method: In each case, try to use the agreement property from §8, applied to the two sides.)

Problem X-22. Show that the Lagrange blending functions $p_i(t)$ for given distinct t_0, \dots, t_n form a basis for the vector space of all polynomials of degree at most n .

(Method #1: Because you already know the vector space has dimension $n + 1$, it is enough to show that the p_i are linearly independent. Suppose some linear combination $c_0 p_0(t) + \dots + c_n p_n(t) = 0$, for all t . Substitute in various appropriate values of t and see if you can show the c_i are all zero.)

(Method #2: It is also enough to show that any f in the vector space is a linear combination of the polynomials $p_i(t)$. Given f , let $c_i = f(t_i)$ and show that $c_0 p_0(t) + \dots + c_n p_n(t) = f(t)$ for $n + 1$ different values of t . Then quote the uniqueness property.)

Problem X-23. This problem shows that a polynomial curve of degree two is a parabola, provided that it does not lie on a straight line.

(a) How can a parabola $y = ax^2 + bx + c$ be expressed parametrically?

(b) Show that a parametric curve of the form $(rt + s, at^2 + bt + c)$ is a parabola, if $r \neq 0$ and $a \neq 0$.

(Method: reparameterize the curve using $u = rt + s$, so $t = \dots$)

(c) What does the graph of the *derivative* of a parametric curve of the form given in (b) look like? Does the derivative curve go through the origin?

(d) Show that if $P(t)$ is a polynomial curve of degree 2 and the graph of $P'(t)$ does not go through the origin, then $P(t)$ is a parabola.

(Method: $P'(t)$ has degree 1, so is a straight line. If $P'(t)$ does not go through the origin, observe that there is a 2×2 rotation matrix R which rotates this line to be parallel to the y -axis. The rotated curve $P(t)R$ has derivative \dots since R is a constant matrix. Integrate to get an expression like that in (b) for $P(t)R$.)

(e) On the other hand, if the graph of $P'(t)$ is a line through the origin, then show that $P(t)$ itself lies on a straight line.

(Method: As in (d), rotate the derivative until its graph is parallel to the y -axis, and then integrate.)

Problem X-24. The curve shown in Figure 3 is actually a polynomial curve of degree 5 that solves a Lagrange problem with $t_i = i$ for each i . With this information, write down an explicit formula for the curve. (Choose a coordinate system with the origin at the lower left data point. Any scale is acceptable. Do not attempt to simplify your answer.)

Problem X-25. A perfect unit circle can be given parametrically by $P(t) = (\cos t, \sin t)$. Suppose $P_n(t)$ is the parametric curve obtained by using instead the Taylor polynomial approximations to $\cos t$ and $\sin t$ of degree at most n . Sketch the curves described by $P_1(t)$ and $P_2(t)$. What do you think happens as for larger and larger values of n ? (Note: Since the Taylor expansion of $\cos t$ has only terms with even powers of t , the Taylor polynomials of degrees, say, 8 and 9 are the same. Similarly, the Taylor polynomials for $\sin t$ of degrees 7 and 8 are the same.)

Problem X-26. Suppose the cubic curve $P(t)$ has $P(0) = P_0$, $P(1) = P_1$, $P(2) = P_2$, and $P(3) = P_3$. Since a cubic curve is determined uniquely by interpolating four points at given t values, everything about $P(t)$ should be describable in terms of P_0, \dots, P_3 .

(a) Give an expression for $P'(0)$ in terms of P_0, \dots, P_3 .

(b) Give an expression for the middle value $P(1.5)$ in terms of P_0, \dots, P_3 .

(Method: First use Lagrange to find an explicit expression for $P(t)$ in terms of the P_i . Are your answers to (a) and (b) linear in the P_i ?)

Problem X-27. Show that every plane polynomial curve of degree k has a mirror symmetry (perhaps not through the origin) for

(a) $k = 1$,

(b) $k = 2$,

(c) $k = 3$.