Sample Final Examination

Problems 1-7 are 10 points each, and short-answer questions (a)-(p) are 3 points each except as noted. Time limit: 3 hours. Useful information: $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{15}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$.

1. Sketch a cubic parametric curve $Q(t)$ such that $Q(0) = (0,0)$, $Q'(0) = (6,0)$, $Q(1) = (0,0)$, $Q'(1) = (0,6)$. Indicate axes and label some points on the axes to indicate the scale. Take care to plot $Q(\frac{1}{2})$ exactly.

2. Find a single matrix, with explicit extries, giving a transformation that takes the box in $\mathbf{R}^3$ with vertices $(0,0,0)$, $(2,0,0)$, $(0,3,0)$, $(0,0,4)$, $(2,3,0)$, $(0,3,4)$, $(2,0,4)$, $(2,3,4)$ to the cube with vertices $(\pm1,\pm1,\pm1)$. (You may choose which vertices go to which.)

3. For the tetrahedron $P = (2,0,0)$, $Q = (0,3,0)$, $R = (0,0,4)$, $S = (1,1,1)$, which faces are visible from above by an orthographic projection, i.e., from the direction $\mathbf{k} = (0,0,1)$? Use a computer method that would be valid for any tetrahedron.

4. The standard tetrahedron (vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$) is projected orthographically on the plane $10x + 10y + 10z = 0$ with up-vector $\mathbf{k} = (0,0,1)$. Find the images of the vertices (as pairs of numbers).

5. Give an equation for the *second* Bézier segment $P^{(2)}(t)$ of the relaxed uniform cubic spline curve in $\mathbf{R}^2$ that interpolates the data points $(3,0)$, $(5,3)$, $(3,5)$, $(0,3)$. (Thus $n = 3$. Your answer should be an explicit equation that could be used on a computer, but it does not have to be simplified algebraically. Treat $P^{(2)}$ as a function on $0 \le t \le 1$.)

6. Suppose $f(t,u)$ is a biquadratic polynomial function that is 0 at all the points (i,j) for $i = 0,1,2$ and $j = 0,1,2$, except that $f(1,1) = 1$. Find $f(\frac{1}{2},\frac{1}{2})$. (There is only one function that fits this description, so if you can think of one, that's it.)

7. Suppose you want to use ray-tracing with Phong shading to make a picture of the surface $z = x^2 - y^2$ on the viewplane $z = 0$. You choose the viewpoint $(0, 0, 4)$ and the lighting direction $\mathbf{s} = (0, 0, 1)$ (in the direction of the viewpoint). How bright should the pixel at $P = (1, 1, 0)$ be? (Notice that the surface is $F(x, y, z) = 0$ for $F(x, y, z) = x^2 - y^2 - z$. Be sure to choose the surface normal direction on the side of the surface towards the viewpoint, so you get a positive brightness.)

Brief-answer questions.

(a) Sketch the points that are convex combinations of the vertices of the standard triangle.

(b) List all 3×3 diagonal matrices that are rotation matrices.

(c) For the relaxed uniform B-spline basis function $f_3(t)$ with $n = 4$, find $f_3(2)$.

(d) For the relaxed uniform interpolating spline basis function $f_2(t)$ with $n = 4$, find $f_2''(0)$.

(e) For the projection on the viewplane $x + y + 2z = 0$ with viewpoint $pt(0, 0, 1, 0)_h$, identify the main classification.

(f) Find the area (in absolute value) of the triangle in $\mathbf{R}^2$ whose vertices are $(8, 64)$, $(9, 81)$, and $(10, 100)$.

(g) Among the line segments $\overline{AB}, \overline{CD}, \overline{EF}$, which pairs cross, if values of affine functions have signs as follows?

function	A	B	C	D	E	F
$f_{AB}()$	0	0	$-$	$+$	$-$	$+$
$f_{CD}()$	$+$	$-$	0	0	$-$	$+$
$f_{EF}()$	$+$	$-$	$+$	$-$	0	0

(h) A certain projective transformation takes the standard unit square to a parallelogram. What is the most you can say about the image of the line at infinity, under this transformation?

(i) Pick four Bézier control points in the answer box at random and indicate how to use the graphical method to calculate $P(\frac{1}{4})$.

(j) For the cubic Bézier curve $P(t)$ in $\mathbf{R}^2$ with control points $(0,0)$, $(1,10)$, $(2,20)$, $(0,0)$, find $P''(0)$ explicitly.

(k) Write down the matrix needed to turn the perspective projection from $(0,0,5)$ into an orthographic projection, with the viewplane being the x,y-plane in both cases.

(l) Write down the matrix needed to turn the oblique projection from the direction $\mathbf{v} = (1,2,3)$ into an orthographic projection, with the viewplane being the x,y-plane in both cases.

(m) For a relaxed, uniform B-spline curve $B(t)$ with $n = 4$, which of the control points $B_0, \ldots, B_4$ affect the value of $B(.23)$?

(n) For the points $A = (2,0)$ and $B = (0,3)$, find $f_{AB}(x,y)$ in the form $ax + by + c$ with explicit a, b, c.

(o) (4 points) Invent an example of a ruled parametric surface (a surface that can be swept out by a moving straight line). Your answer should be some kind of explicit formula for a function $P(t,u)$.

(p) (4 points) For the parametric bilinear patch with data points P_{ij}, $i = 0,1$, $j = 0,1$, find the normal vector for the point where $t = 0$, $u = 0$. A unit normal is not required.