Affine transformations, Part II

1. Mapping one triangle to another with an affine transformation

First, consider the *standard triangle* in $\mathbf{R}^2$ whose vertices are $(1,0)$, $(0,1)$, $(0,0)$, which we can call $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{0}$. Let PQR be any triangle.

Problem 1.1 . Find the extended matrix of an affine transformation T that takes the standard triangle to the triangle PQR (with $T(\mathbf{e}^{(1)}) = P$, etc.). (See Figure 1.)

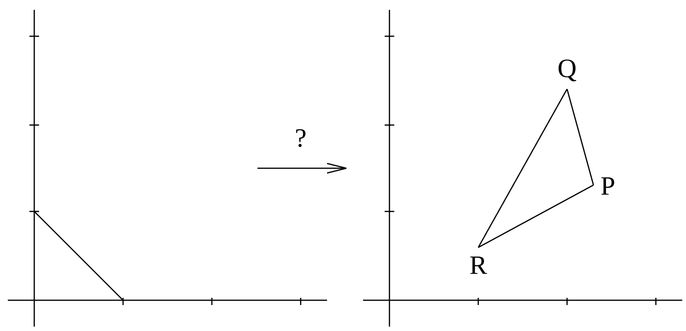

Figure 1: Standard triangle to a given triangle.

Solution. Let the triangle $P'Q'\mathbf{0}$ be the result of translating the triangle PQR so that R goes to the origin, by subtracting R from all three vertices. Thus $P' = P - R$ and $Q' = Q - R$. The desired affine transformation consists of (1) a homogeneous linear transformation taking $\mathbf{e}^{(1)}$ to P' and $\mathbf{e}^{(2)}$ to Q', followed by (2) translation by R. Because $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are the standard basis vectors, the rows of the desired 2×2 matrix for (1) are just their images. Putting (1) and (2) together, we see that the desired transformation has extended matrix $\begin{bmatrix} (P-R) & 0 \\ (Q-R) & 0 \\ R & 1 \end{bmatrix}$ (to use an obvious shorthand notation). (See Figure 2.)

For example, to map the standard triangle to the triangle with vertices $(2,1), (4,3), (2.5,4)$, the extended matrix is $\begin{bmatrix} -.5 & -3 & 0 \\ 1.5 & -1 & 0 \\ 2.5 & 4 & 1 \end{bmatrix}$.

(To check that this matrix works, try multiplying the extended vertices $(1,0,1)$, $(0,1,1)$, $(0,0,1)$ by it.)

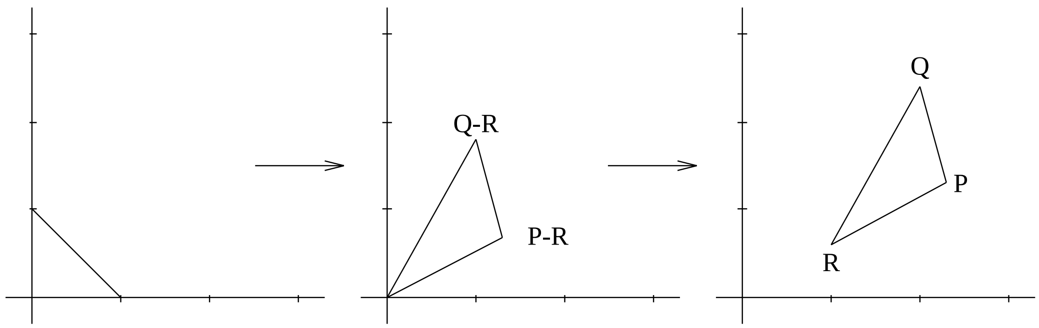

Figure 2: Standard triangle to a given triangle: solution

Problem 1.2 . Find how to calculate the extended matrix of an affine transformation U that takes the triangle PQR to the triangle $P'Q'R'$ (with $U(P) = P'$, etc.). (See Figure 3.)

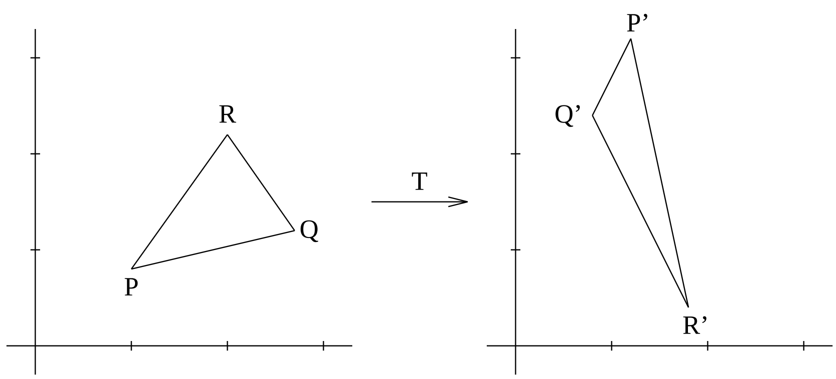

Figure 3: Arbitrary triangle to arbitrary triangle

Solution. Use the standard triangle as an intermediate stage. You know how to find T taking the standard triangle to PQR and T' taking the standard triangle to $P'Q'R'$. T^{-1} takes PQR back to the standard triangle. Therefore $U = T^{-1}T'$, so that U has extended matrix

$$\begin{bmatrix} (P-R) & 0 \\ (Q-R) & 0 \\ R & 1 \end{bmatrix}^{-1} \begin{bmatrix} (P'-R') & 0 \\ (Q'-R') & 0 \\ R' & 1 \end{bmatrix}.$$

(For a specific example you could put in numbers and multiply out. If P, Q, R are noncollinear, so that PQR is a genuine triangle, then the left matrix will be invertible as required. Computationally, one way to calculate a product $A^{-1}B$ is to write down an extended matrix $[A|B]$ and row-reduce it to get a matrix $[I|C]$; here C will be $A^{-1}B$.)

Note. In $\mathbf{R}^3$, you can use the same sort of method to move a tetrahedron to a tetrahedron.

2. Mapping one parallelogram to another

It is a fact that an affine transformation takes lines to lines parallel lines to parallel lines (all provided that it is nonsingular). Therefore nonsingular affine transformations in $\mathbf{R}^2$ take parallelograms to parallelograms.

Problem 2.1 . In $\mathbf{R}^2$, is it possible to take an *arbitrary* parallelogram $PQRS$ to an *arbitrary* parallelogram $P'Q'R'S'$ using an affine transformation? (See Figure 4.)

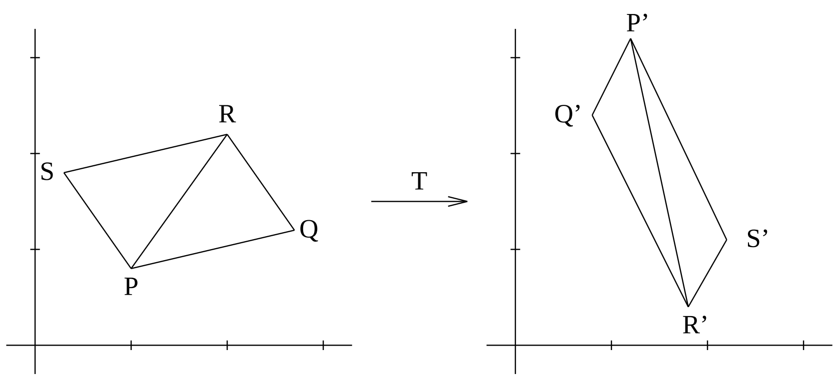

Figure 4: Mapping parallelograms

Solution. Yes: Just find T taking the triangle PQR to the triangle $P'Q'R'$. In order for the image of the parallelogram to *be* a parallelogram, $T(S)$ will automatically be the point S', as desired.

You can think of this solution in two steps: Taking the first parallelogram to the "standard square" and taking the "standard square" to the second parallelogram. Of course, either parallelogram or both could actually be a rectangle or even a square.

3. The hidden explanation of extended matrices

The extended matrix for an affine transformation in $\mathbf{R}^2$ is a 3×3 matrix. At first extended matrices may seem to be just an algebraic trick, but there is actually a geometrical interpretation.

The basic idea is simple: A homogeneous linear transformation can't move the origin. However, if you "embed" $\mathbf{R}^2$ as the $z=1$ plane in $\mathbf{R}^3$ and consider the homogeneous linear transformations on $\mathbf{R}^3$ that take the $z=1$ plane only to itself, these homogeneous linear transformations *can* duplicate the effect of affine transformations on the $z = 1$ plane. The 3×3 matrix you need is precisely the extended matrix of the affine transformation. There is no problem about moving the origin, because the "origin" of the $z = 1$ plane is not the origin of $\mathbf{R}^3$. This three-dimensional interpretation is normally "hidden", in that you don't have to think about it when you use extended matrices.

Figure 5 illustrates how the idea works for the translation $T(\mathbf{x}) = \mathbf{x} + (2, 1)$, which has extended matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$. The transformation on $\mathbf{R}^3 \to \mathbf{R}^3$ takes the $z = 1$ plane to itself and in that plane produces the desired translation.

4. Problems

Problem I-1. Suppose that A is a 3×3 matrix with the property that $(x, y, 1)A$ has the form $(\cdot, \cdot, 1)$, no matter what the numbers x and y are. Show that the entries of the third column of A *must* be $0, 0, 1$.

Problem I-2. Let $P = (2, -2)$, $Q = (4, 1)$, $R = (3, 0)$.

(a) Find an affine transformation that moves the standard triangle to the triangle PQR (with $T((1, 0)) = P$, etc.).

(b) Find an affine transformation that moves PQR to the standard triangle.

(c) Find an affine transformation that takes P, Q, R to P, R, Q respectively.

Problem I-3. (a) Show that if T is some sort of transformation obeying the law $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, then $T(\mathbf{0}) = \mathbf{0}$. (Therefore an affine transformation does *not* obey this law unless it happens to be a homogeneous linear transformation.)

(b) Show that if T is an affine transformation, then T *does* obey the law $T(\mathbf{x} - \mathbf{y} + \mathbf{z}) = T(\mathbf{x}) - T(\mathbf{y}) + T(\mathbf{z})$. (Start by writing $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$.)

Problem I-4. Let $PQRS$ be a parallelogram, as pictured in Section **2** above. It is an easy fact that S can be expressed algebraically in terms of P, Q, R as $S = P - Q + R$. Use this fact and the result of Problem I-3 to derive algebraically the statement made in §2 above that if T is an affine transformation taking three consecutive vertices of a parallelogram to three consecutive vertices of another parallelogram, then T takes the fourth vertex of the first parallelogram to the fourth vertex of the second. (Here vertices P, Q, R are called "consecutive" if $\overline{PQ}$ and $\overline{QR}$ are edges of the parallelogram.)

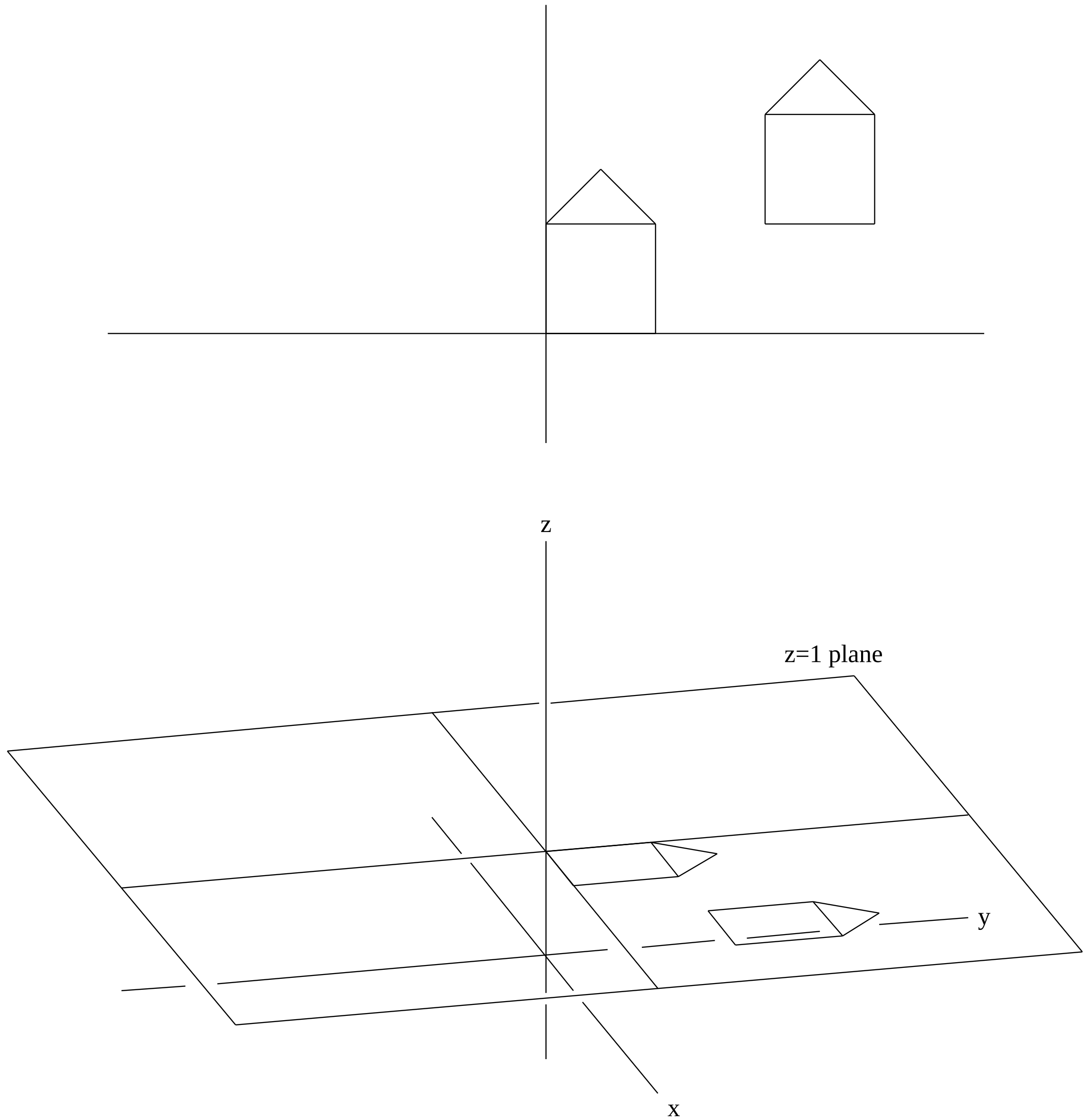

Figure 5: The hidden explanation.

Problem I-5. Let P, Q, R be three points in $\mathbf{R}^2$, and let T be the affine transformation that takes the standard triangle to the triangle PQR.

(a) What is the area of the standard triangle in $\mathbf{R}^2$?

(b) An interesting quantity Δ associated with a triangle PQR has several equivalent expressions:

$$\boxed{\Delta = \det \begin{bmatrix} (P-R) \\ (Q-R) \end{bmatrix} = \det \begin{bmatrix} (P-R) & 0 \\ (Q-R) & 0 \\ R & 1 \end{bmatrix} = \det \begin{bmatrix} P & 1 \\ Q & 1 \\ R & 1 \end{bmatrix}}$$

Show that these expressions do in fact have the same value, so that any one of them can be used to define Δ. (Method: Use facts about determinant expansion and row operations.)

(c) Show that the area of a triangle PQR is given by $\boxed{\text{area} = \frac{1}{2}|\Delta|}$.

(Method: Think about the affine transformation T taking the standard triangle to PQR, as in Problem I-1.1. This transformation is constructed as the composition of a homogeneous linear transformation and a translation. By what factor does each change areas?)

(d) Explain why the sign of Δ tells whether you are going counterclockwise or clockwise if you go from P to Q to R and back to P. (Method: As in (c). Notice that for the standard triangle the vertices are listed going counterclockwise.)

(e) For the specific example $P = (1, 2)$, $Q = (3, 3)$, $R = (2, 4)$, find the area of the triangle PQR and say from Δ whether PQR are listed going counterclockwise or clockwise.

(f) If $\Delta = 0$, what does that say about how the points P, Q, R lie with respect to one another?

Problem I-6. As you know, the graph of a function $f : \mathbf{R} \to \mathbf{R}$ is a set of points in the plane, namely $(x, y) \in \mathbf{R}^2 : y = f(x)$. More generally, the graph of any function between sets, $f : X \to Y$, is the set $(x, y) \in X \times Y : y = f(x)$. In this definition, recall that $X \times Y$ just means the set of all pairs (x, y) with $x \in X, y \in Y$.

(a) Show that a function $T : \mathbf{R}^m \to \mathbf{R}^n$ is a homogeneous linear transformation if and only if the graph of T is a subspace of $\mathbf{R}^m \times \mathbf{R}^n$, or equivalently, of $\mathbf{R}^{m+n}$.

(This is a $\Leftrightarrow$ problem; show each direction separately. Do the $\Rightarrow$ direction first. In the $\Leftarrow$ direction, notice that to start with you are not assuming that T is a homogeneous linear transformation; it's just some function with a graph that is a subspace, and you must verify that the defining properties of a homogeneous linear transformation hold.)

(b) Show that a function $T : \mathbf{R}^m \to \mathbf{R}^n$ is an affine transformation if and only if its graph is flat, meaning that its graph is a subspace translated by some constant vector. (For the constant vector you can use $T(\mathbf{0})$.)

Problem I-7. Suppose that a graphics package uses device coordinates (i.e., screen coordinates) in which the *upper left* corner of the screen is the origin and the *lower right* corner is $(1024, 768)$. Suppose that you want to plot points in a window that in world coordinates is expressed by $-2 \le x \le 6$ and $-1 \le y \le 5$. What affine transformation transforms world-coordinate pairs to device-coordinate pairs? Solve this problem by using a general method for finding an affine transformation that takes one given parallelogram to another.

Problem I-8. An older laser printer has a graphics mode with device coordinates in which the *lower left* corner of the screen is the origin and the *upper right* corner is $(1023, 780)$. Suppose someones asks you to plot data pairs from a window that is taller than it is wide, say with $0 \le x \le 3$ and $0 \le y \le 4$. To accomplish this, you want to transform the data pairs so that the user origin is at the upper left corner of the screen and the graph is on its side; of course, after the graph is plotted on the Imagen you can turn the paper any way you like. By what affine transformation should you transform the data pairs before sending the data to the printer? (Be sure that the graph does not come out non-uniformly scaled or reflected. A window that is wider than it is tall is said to have *landscape* shape; a window that is taller than it is wide is said to have *portrait* shape.)

Problem I-9. PostScript printers assume an origin at the lower left corner of the output paper, if you're holding at the paper with the longer direction being up and down ("portrait position"). The assumed unit length is one "point" ($= 1/72$ of an inch). Suppose you want to print a picture that is wider than it is tall ("landscape position"), with the origin also at the lower left, using the whole sheet of 8.5×11-inch paper. One way would be to transform your output points by an affine transformation so that the picture is rotated by $90°$ and the origin is moved to the lower right-hand corner. Give the extended matrix of such a transformation. (Remember that your transformation is not in inches. Actually, in PostScript you can simply declare a new coordinate system in terms of the old one, but the same matrix is needed to do this. In this problem, which position is in user coordinates and which in device coordinates?)

Problem I-10. Suppose you have a program that plots on a screen in Tektronix coordinates (lower left $(0,0)$, upper right $(1023,780)$). You then want to show four plots on the screen at once, each half as large in each dimension, by dividing the screen into four rectangles of the same shape (four *viewports*). Give four affine transformations, which, if applied to the output of your program, will put the results in the four viewports. For example, the first transformation could transform the whole screen to the upper left quarter-screen.

Problem I-11. In $\mathbf{R}^3$, by the *standard tetrahedron* let us mean the tetrahedron with vertices $\mathbf{e}^{(1)} = (1,0,0)$, $\mathbf{e}^{(2)} = (0,1,0)$, $\mathbf{e}^{(3)} = (0,0,1)$, $\mathbf{0} = (0,0,0)$, listed in that order.

(a) Find the volume of the standard tetrahedron. (From calculus, the volume of a tetrahedron is 1/3 times the area of the base times the height.)

(b) Using the geometrical interpretation of the determinant and the two-step construction of T, analogously to Problem I-1.1, explain why the volume of a tetrahedron $PQRS$ is given by $\boxed{\text{volume} = \frac{1}{6}|\Delta|}$ where $\Delta = \det \begin{bmatrix} (P-S) \\ (Q-S) \\ (R-S) \end{bmatrix}$.

(c) By using facts about determinant expansion and and row operations, show that Δ can be written either as the determinant of the *extended* matrix for T or even more symmetrically:

$$\boxed{\Delta = \det \begin{bmatrix} (P-S) & 0 \\ (Q-S) & 0 \\ (R-S) & 0 \\ S & 1 \end{bmatrix} = \det \begin{bmatrix} P & 1 \\ Q & 1 \\ R & 1 \\ S & 1 \end{bmatrix}.}$$

(d) Let $P = (5,7,1)$, $Q = (4,2,0)$, $R = (8,11,-3)$, $S = (7,9,-3)$. Find the volume of the tetrahedron $PQRS$.

(e) Can you find a geometrical interpretation of the sign of Δ?

Problem I-12. (a) The *standard square* in $\mathbf{R}^2$ is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Find an affine transformation that takes the standard square to the square $PQRS$, with $(0,0)$ going to P and $(1,0)$ going to Q, where $P = (5,3)$, $Q = (9,4)$, and the vertices are listed counterclockwise. (Method: You could try to find R and use triangle methods, but it is simpler to combine a homogeneous linear transformation of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with a translation.)

(b) A former student from this class, now in industry, has asked how to find an affine transformation on $\mathbf{R}^2$ that preserves all shapes (but not size) and takes a given line segment $\overline{PQ}$ to another given line segment $\overline{P'Q'}$. Describe such a method.

Problem I-13. In $\mathbf{R}^2$, consider the standard square (mentioned in Problem I-12). Write down matrices for all eight affine linear transformations that take this square to itself rigidly, not necessarily leaving vertices fixed.

Problem I-14. Suppose you are given a 2×2 matrix A. The homogeneous linear transformation $T(\mathbf{x}) = \mathbf{x}A$ of course leaves the origin fixed. Suppose you want an affine transformation U that is like T except that it leaves a certain point P_0 fixed instead of the origin. It makes sense to let U be of the form $U(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$, for the same A and for some suitable vector $\mathbf{b}$. But what $\mathbf{b}$ should you use to make $U(P_0) = P_0$? (Your answer should express $\mathbf{b}$ in terms of A and P_0.)

Problem I-15. Suppose PQR and $P'Q', R'$ are two congruent triangles in $\mathbf{R}^3$. Say explicitly how to find an affine transformation that takes the first triangle to the second with a rigid motion. (A detailed proof is not required.)

(Suggested method: In $\mathbf{R}^3$, the analogue of the triangle method in $\mathbf{R}^2$ maps tetrahedra, not triangles. You could make tetrahedra just by choosing arbitrarily a fourth vertex to go with each triangle, but the resulting affine transformation is not necessarily rigid. What you need is a way of choosing a fourth vertex for each triangle in a regular way so that the two resulting tetrahedra are congruent. Then the motion will be rigid on the tetrahedra and so, you may assume, on all of $\mathbf{R}^3$. One way: Find the fourth vertices S and S' by taking the cross product of two sides of each triangle and adding to a vertex. Your answer may be given as a product $P^{-1}Q$ with entries of P and Q clearly stated. Interestingly, P and Q individually may not give rigid motions, but $P^{-1}Q$ will.)

Problem I-16. This is a discussion of how to make the smoke in Picture #1 of a house in Handout C, by using an idea that occurs frequently in this course: Make drawings in strange positions by first making easy figures in easy positions and then applying affine transformations.

To make an easy wavy graph somewhat like the smoke, consider the graph of $y = x \sin x$ for $0 \le x \le 8\pi$. Since $\sin x$ has values bouncing between -1 and 1, $x \sin x$ has values bouncing between the line $y = x$ and the line $y = -x$. This same graph can be expressed parametrically as $(x, y) = (t, t \sin t)$ for $0 \le t \le 8\pi$; parametric curves are more easily mapped.

The problem: Find an affine transformation that when applied to this graph gives the smoke in the correct position. If you wish, you may express your answer as a product of matrices and inverses of matrices, each with explicit entries.

One possible method: Think of the curve as drawn in the triangle PQR, where $P = (8\pi, -8\pi)$, $Q = (8\pi, 8\pi)$, and R is the origin. Transform the triangle several times to get it to be the correct size and in the correct position.

Or another possible method: Using the triangle-to-triangle method of Problem 1.2, transform the triangle just mentioned to a triangle containing the smoke, whose base goes from $(\frac{5}{3}, \frac{5}{2})$ to $(\frac{11}{6}, \frac{5}{2})$ (the top of the chimney) and whose top vertex is $(\frac{7}{4}, \frac{17}{6})$.

Problem I-17. Suppose you want to have a computer imitate handwritten letters. One good approach is to design "model letters" in a rectangle and then apply transformations to reshape them a bit as desired. For example, your model lower-case "d" might consist of a circle of radius $\frac{1}{2}$ and center $(\frac{1}{2}, \frac{1}{2})$, together with the line segment from $(1, 0)$ to $(1, 2)$.

(a) Give a transformation that would turn the model "d" into a thinner letter, in the rectangle $2 \le x \le 2.75$, $1 \le y \le 3$.

(b) Give a specific example of a transformation that would turn the model "d" into a version that is slanted and thinner but of the same height, somewhat like italic "*d*" but without the curl at the lower right. (You may choose where to position the image.)

Problem I-18. In $\mathbf{R}^2$ a line is like a one-dimensional subspace, except that a subspace goes through the origin and a line does not have to. You could say that a line is just a one-dimensional subspace that has been translated. The same is true of lines in $\mathbf{R}^3$. Planes in $\mathbf{R}^3$ could be described as two-dimensional subspaces that have been translated. More generally, in $\mathbf{R}^n$, a *flat* is any set that can be obtained by translating a subspace. Thus lines and planes are examples of flats. In $\mathbf{R}^n$, for any $k \le n$ a *k-simplex* is a set of $k+1$ points that does not lie in a $(k-1)$-dimensional flat.

(a) In $\mathbf{R}^3$, what is a more common name for a 2-simplex? For a 3-simplex? For a 1-simplex?

(b) State without proof a fact about the possibility of mapping one n-simplex to another in $\mathbf{R}^n$.

Problem I-19. (a) Which homogeneous linear transformations on $\mathbf{R}^2 \to \mathbf{R}^2$ take the line $y = 1$ to itself, i.e., take points of the form $(a, 1)$ to points of the same form $(\ldots, 1)$? (b) Which homogeneous linear transformations on $\mathbf{R}^3 \to \mathbf{R}^3$ take the plane $z = 1$ to itself? (In each case, give your reasoning; the answer should describe the possible matrices precisely.)

Problem I-20. (a) Show that the formula for the matrix that maps one triangle onto another can be written more symmetrically:

$$\begin{bmatrix} (P-R) & 0 \\ (Q-R) & 0 \\ R & 1 \end{bmatrix}^{-1} \begin{bmatrix} (P'-R') & 0 \\ (Q'-R') & 0 \\ R' & 1 \end{bmatrix} = \begin{bmatrix} P & 1 \\ Q & 1 \\ R & 1 \end{bmatrix}^{-1} \begin{bmatrix} P' & 1 \\ Q' & 1 \\ R' & 1 \end{bmatrix}.$$

(Method: $\begin{bmatrix} (P-R) & 0 \\ (Q-R) & 0 \\ R & 1 \end{bmatrix}$ can be changed into $\begin{bmatrix} P & 1 \\ Q & 1 \\ R & 1 \end{bmatrix}$ by row operations, and row operations can be achieved by pre-multiplying by a suitable elementary matrix E, or in other words, a matrix that is like the identity matrix except for some entries in one column or one row. Find E explicitly. The right side of the matrix equation then has the form $(E\hat{A})^{-1}(E\hat{A}')$. Simplify.)

(b) Of course, all this only works for an expression of the form $()^{-1}()$. We cannot use $E\hat{A}$ in place of A generally. Show that even so, $\det(E\hat{A}) = \det \hat{A}$.

(This fact was behind earlier problems for formulas for the area of a triangle and volume of a tetrahedron. Method: Use the fact that determinants are compatible with matrix multiplication.)

Problem I-21. In $\mathbf{R}^2$, suppose O, P, Q, S are vertices of a square obtained by rotating the standard square by R_{45°, so that $P = (1,0)R_{45^\circ}$, that $Q = (1,1)R_{45^\circ}$, and that $S = (0,1)R_{45^\circ}$. Find a matrix for an affine transformation taking O, P, Q, S to the parallelogram $(1,0), (2,0), (3,1), (2,1)$. You may leave your answer as a product of matrices with explicit entries.

Problem I-22. (a) Using calculus, verify that the region under the parabola $y = 1 - x^2$ takes up $\frac{2}{3}$ of the area of the rectangle with $-1 \le x \le 1$, $0 \le y \le 1$.

(b) A fact due to Archimedes is that a parabola inscribed in a rectangle, from one vertex to an adjacent vertex and tangent to the opposite side, bounds a region with $\frac{2}{3}$ of the area of the rectangle. Use (a) and a transformation to prove this statement. (Assume without proof the fact that each rectangle with two specified adjacent vertices contains only one inscribed parabola through them.)

Problem I-23. A certain device has a window with lower left coordinates $(u_{\mathrm{lo}}, v_{\mathrm{lo}})$ and upper right coordinates $(u_{\mathrm{hi}}, v_{\mathrm{hi}})$, and you need to write a routine to map to it an arbitrarily specified user window with lower left coordinates $(x_{\mathrm{lo}}, y_{\mathrm{lo}})$ and upper right coordinates $(x_{\mathrm{hi}}, y_{\mathrm{hi}})$. Explain how to map the windows in each of these cases:

(a) The user window is to be mapped onto the device window nonuniformly (i.e., the horizontal and vertical coordinates work independently). Your answer will be a 3×3 matrix whose homogeneous part is diagonal and whose entries are formulas involving the corner data given.

(b) The user window is to be mapped into the device window uniformly (so the horizontal and vertical axes are scaled by the same factor). If the two windows are not the same shape, then it will not be possible to map one window *onto* the other. Instead, transform the user window so that its image fills the device window horizontally and is centered vertically, or vice-versa. Your answer will be a 3×3 matrix whose homogeneous part is a 2×2 scalar matrix and whose entries depend on the corner data. Say how to compute these entries. You will need either to break the solution into cases or else to use `max` or `min` functions.